PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 102, Number 4, April 1988

POLYNOMIAL-RATIONAL BIJECTIONS OF $\mathbf{R}^n$

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(Communicated by Paul S. Muhly)

ABSTRACT. It is shown in this note that every invertible polynomial transformation of $\mathbf{R}^n$ of degree two has a rational inverse defined on the whole space $\mathbf{R}^n$. The same is true for polynomial transformations of higher degrees, satisfying some differential condition which is a real analogue of Jagžev's condition considered in [**3**, **4**, and **6**].

The proofs of these statements are based on the Białynicki-Birula and Rosenlicht surjectivity theorem [**2**] and on standard properties of complex dominant polynomial mappings.

1. Introduction. It was proved by Białynicki-Birula and Rosenlicht [**2**] that every injective polynomial transformation of $\mathbf{R}^n$ is necessarily surjective and thus, by the Brouwer theorem on invariance of domain, is a homeomorphism of $\mathbf{R}^n$ onto $\mathbf{R}^n$. The inverse of such a mapping is not in general a polynomial nor even a rational mapping.

It is well known that the inverse of a bijective polynomial mapping $F\colon \mathbf{R}^n \to \mathbf{R}^n$ is a polynomial if and only if the Jacobian of F is invertible in the ring $\mathbf{R}[x_1, \ldots, x_n]$; i.e. the polynomial $\operatorname{Jac} F = \det F'(x)$ is a nonzero constant (see, e.g., [**1**, Theorem 2.1]).

Note that the Jacobian of a polynomial bijection of $\mathbf{R}^n$ does not change its sign on $\mathbf{R}^n$, i.e. it is either nonnegative or nonpositive everywhere. This can be easily verified by using the notion of the topological degree of a C^1-mapping (see, for example, [**7**, Chapter III.B]). Indeed, suppose that $F\colon \mathbf{R}^n \to \mathbf{R}^n$ is a polynomial homeomorphism such that $\operatorname{Jac} F(a) > 0$ and $\operatorname{Jac} F(b) < 0$ for two different points $a, b \in \mathbf{R}^n$. Let D denote an open ball which contains both points a and b. Then $\deg(F(a), F, D) = 1$ and $\deg(F(b), F, D) = -1$. This contradicts the well-known property of the topological degree of F which asserts that it is constant on every connected component of the set $\mathbf{R}^n \backslash F(\partial D)$.

It is still unknown ("the real Jacobian Conjecture") whether for a polynomial mapping $F\colon \mathbf{R}^n \to \mathbf{R}^n$ the assumption that $\operatorname{Jac} F \neq 0$ on $\mathbf{R}^n$ ("the real Jacobian Condition") implies the injectivity of F.

Note that the inverse of a polynomial bijection of $\mathbf{R}^n$ satisfying the real Jacobian Condition may not be even a rational mapping. (As a simple counterexample, one can take $F\colon \mathbf{R} \ni x \to x^3 \in \mathbf{R}$.)

It will be shown in §3 that for a polynomial transformation of $\mathbf{R}^n$ of degree two the real Jacobian Condition implies bijectivity as well as rationality of the inverse.

In case of degrees equal to or greater than three, both these properties are guaranteed by some effectively verifiable differential condition, more restrictive than

Received by the editors December 1, 1986.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 14E05; Secondary 12D99.

the real Jacobian Condition. This condition is related to that proposed by Jagžev [**4**] (see also [**3** and **6**]).

We are very grateful to L. M. Drużkowski for his helpful remarks.

2. Polynomial transformations of arbitrary degree. Let us first recall some standard facts from complex algebraic geometry.

Namely, if $F = (F_1, \dots, F_n)$: $\mathbf{C}^n \to \mathbf{C}^n$ is a *dominant polynomial mapping* (i.e. if $F(\mathbf{C}^n)$ is dense in $\mathbf{C}^n$) then the field $\mathbf{C}(x_1, \dots, x_n)$ of rational functions is a finite extension of the field $\mathbf{C}(F_1, \dots, F_n)$. We have

$$d(F) = [\mathbf{C}(x_1, \dots, x_n): \mathbf{C}(F_1, \dots, F_n)] = \max\{\#F^{-1}(y): y \in \mathbf{C}^n, \#F^{-1}(y) < \infty\},$$

where $\#F^{-1}(y)$ denotes the cardinality of the set $F^{-1}(y)$. Moreover, the set $S(F) = \{y \in \mathbf{C}^n: \#F^{-1}(y) \neq d(F)\}$ is a nowhere dense algebraic subset of $\mathbf{C}^n$ (cf. [**5**, Proposition 3.17]).

For a polynomial mapping F: $\mathbf{R}^n \to \mathbf{R}^n$, we define *the complexification of* F as the unique polynomial mapping $\tilde{F}$: $\mathbf{C}^n = \mathbf{R}^n + i\mathbf{R}^n \to \mathbf{C}^n = \mathbf{R}^n + i\mathbf{R}^n$ such that $\tilde{F} = F$ on $\mathbf{R}^n = \mathbf{R}^n + i0$.

This definition is right because the set $\mathbf{R}^n \subset \mathbf{C}^n$ is an identity set for complex polynomials, i.e. if $f \in \mathbf{C}[x_1, \dots, x_n]$ and $f = 0$ on $\mathbf{R}^n$, then $f = 0$ on $\mathbf{C}^n$. A useful equivalent formulation of this identity principle asserts that if $X \subset \mathbf{C}^n$ is a complex algebraic set and $\mathbf{R}^n \subset X$, then $X = \mathbf{C}^n$.

In addition to the earlier mentioned surjectivity theorem by Białynicki-Birula and Rosenlicht [**2**] we also need the following result.

PROPOSITION 2.1. *Let* F: $\mathbf{R}^n \to \mathbf{R}^n$ *be a bijective polynomial mapping such that* $\tilde{F}^{-1}(\mathbf{R}^n) = \mathbf{R}^n$. *Then the mapping* F^{-1}: $\mathbf{R}^n \to \mathbf{R}^n$ *is rational.*

PROOF. According to Chevalley's theorem (cf. [**5**, Proposition 2.31]), the set $\tilde{F}(\mathbf{C}^n)$ is constructible and, by the assumption, it contains $\mathbf{R}^n$. Therefore, the standard closure of $\tilde{F}(\mathbf{C}^n)$ is an algebraic set (cf. [**5**, Theorem 2.33]) which contains $\mathbf{R}^n$, so it coincides with $\mathbf{C}^n$ in view of the above identity principle. This proves that the mapping $\tilde{F}$: $\mathbf{C}^n \to \mathbf{C}^n$ is dominant.

Suppose that $d(\tilde{F}) > 1$. Then, by the assumptions, $\mathbf{R}^n \subset S(\tilde{F}) \neq \mathbf{C}^n$ which is impossible in view of the same identity principle. Therefore, $d(\tilde{F}) = 1$, i.e. $\mathbf{C}(x_1, \dots, x_n) = \mathbf{C}(\tilde{F}_1, \dots, \tilde{F}_n)$, and

$$S(\tilde{F}) = (\mathbf{C}^n \setminus \tilde{F}(\mathbf{C}^n)) \cup \{y \in \mathbf{C}^n: \#\tilde{F}^{-1}(y) = \infty\}.$$

Consider the rational inverse of the mapping $\tilde{F}$, namely $\tilde{F}^{-1}$: $\mathbf{C}^n \setminus S(\tilde{F}) \to \mathbf{C}^n$. Since $S(\tilde{F}) \cap \mathbf{R}^n = \emptyset$, the mapping F^{-1} is the restriction of the mapping $\tilde{F}^{-1}$ to $\mathbf{R}^n$ and hence it is also rational. Q.E.D.

In the sequel, a bijective polynomial transformation of $\mathbf{R}^n$ with rational inverse will be called a *polynomial-rational bijection of* $\mathbf{R}^n$.

To explain the special role of polynomials of degree three we recall, in a version convenient for our purpose, Jagžev's result [**4**] (see also [**1** and **3**]) on the reduction of degree.

PROPOSITION 2.2. *Let* F: $\mathbf{R}^n \to \mathbf{R}^n$ *be a polynomial mapping such that* $\deg F \geq 3$ *and* $\operatorname{Jac} F \neq 0$ *in* $\mathbf{R}[x_1, \dots, x_n]$. *Then there exists a smallest positive integer* $N = N(n, \deg F) \geq n$ *and a unique polynomial mapping* $F_{\text{red}} = I + H$: $\mathbf{R}^N \to \mathbf{R}^N$, *where* I *is the identity mapping and* H *is a homogeneous polynomial of degree* *three, such that F is a polynomial-rational bijection of* $\mathbf{R}^n$ *if and only if* F_{red} *is a polynomial-rational bijection of* $\mathbf{R}^N$.

It is worthwhile adding that both N andF_{red} can be effectively determined (see [**3**]).

Let H: $\mathbf{R}^n \to \mathbf{R}^n$ be a homogeneous polynomial of positive degree k. It is well known that there exists a unique symmetric k-linear mapping $\hat{H}$: $(\mathbf{R}^n)^k \to \mathbf{R}^n$ such that $\hat{H}(x, \dots, x) = H(x)$ for every $x \in \mathbf{R}^n$.

For example, if $\deg H = 2$, then

$$\hat{H}(x,y) = \tfrac{1}{2}(H(x+y) - H(x) - H(y)) \quad \text{for } x, y \in \mathbf{R}^n,$$

and if $\deg H = 3$, then

$$\hat{H}(x,y,z) = \tfrac{1}{6}(H(x+y+z) - H(x+y) - H(x+z) - H(y+z) + H(x) + H(y) + H(z)) \quad \text{for all } x, y, z \in \mathbf{R}^n.$$

The basic notion of this section is the following counterpart of Jagžev's condition [**4**] (see also [**3**, **6**]).

DEFINITION 2.3. We say that a polynomial mapping $F = I + H$: $\mathbf{R}^n \to \mathbf{R}^n$, where H is a homogeneous polynomial of degree three, satisfies condition $(\mathrm{J_r})$ if

$$\det(I + \hat{H}(x,x,\cdot) + \hat{H}(y,y,\cdot)) \det(I + \hat{H}(x,x,\cdot) - \hat{H}(y,y,\cdot)) \neq 0$$

for every $x, y \in \mathbf{R}^n$. (We denote by $\hat{H}(a,b,\cdot)$ the linear mapping $\mathbf{R}^n \ni x \to \hat{H}(a,b,x) \in \mathbf{R}^n$.)

Since for a mapping $F = I + H$ we have $F'(x) = I + \hat{H}(\sqrt{3}x, \sqrt{3}x, \cdot)$ for every $x \in \mathbf{R}^n$, so condition $(\mathrm{J_r})$ implies the real Jacobian Condition.

The role of condition $(\mathrm{J_r})$ in the examinations of polynomial-rational bijections explains

PROPOSITION 2.4. *Let H: $\mathbf{R}^n \to \mathbf{R}^n$ be a homogeneous polynomial of degree three such that the mapping $F = I + H$ satisfies condition* $(\mathrm{J_r})$. *Then F is a polynomial-rational bijection of* $\mathbf{R}^n$.

PROOF. By the three-linearity of $\hat{H}$ we obtain the following formula:

$$F(x) - F(y) = (I + \hat{H}(a(x,y), a(x,y), \cdot) + \hat{H}(b(x,y), b(x,y), \cdot))(x - y),$$

where $a(x,y) = \sqrt{2}/2x + (\sqrt{2} + \sqrt{6})/4y$ and $b(x,y) = \sqrt{2}/2x + (\sqrt{2} - \sqrt{6})/4y$ for all $x, y \in \mathbf{R}^n$. Because of condition $(\mathrm{J_r})$ we see that the mapping F is injective. Applying the surjectivity theorem [**2**] which is mentioned at the beginning of the Introduction, we obtain the bijectivity of F.

The complexification of the mapping F has the form:

$$\tilde{F}(x+iy) = x + H(x) - 3\hat{H}(x,y,y) + i(y - H(y) + 3\hat{H}(x,x,y)) \quad \text{for every } x, y \in \mathbf{R}^n.$$

If $\tilde{F}(x+iy) \in \mathbf{R}^n$, then $(I + \hat{H}(\sqrt{3}x, \sqrt{3}x, \cdot) - \hat{H}(y,y,\cdot))(y) = 0$. By condition $(\mathrm{J_r})$ the linear mapping $I + \hat{H}(\sqrt{3}x, \sqrt{3}x, \cdot) - \hat{H}(y,y,\cdot)$ is an isomorphism. Thus $y = 0$, i.e. $\tilde{F}^{-1}(\mathbf{R}^n) = \mathbf{R}^n$. We conclude the proof applying Proposition 2.1. Q.E.D.

Let us now give an example of a bijective polynomial transformation with polynomial inverse without the property $(\mathrm{J_r})$.

EXAMPLE 2.5 (cf. [**6**, Proposition 2.2]). Let $F = I + H\colon \mathbf{R}^5 \to \mathbf{R}^5$, where

$$H(x_1, x_2, x_3, x_4, x_5) = (0, 0, x_1x_2x_4 + x_1^2x_5, -x_1x_2x_3 + x_1^2x_5, \tfrac{1}{2}x_2^2(x_3 + x_4)).$$

Then $\operatorname{Jac} F = 1$ and F is a bijection of $\mathbf{R}^5$ with polynomial inverse. However, if $u = (\sqrt{3/2}, \sqrt{3/2}, 0, 0, 0)$, $v = (\sqrt{3/2}, -\sqrt{3/2}, 0, 0, 0)$, and $w = (0, 0, 1, 1, -1)$, then $w + \hat{H}(u, u, w) + \hat{H}(v, v, w) = 0$, i.e. the mapping F does not satisfy condition $(\mathrm{J_r})$.

As an immediate consequence of Propositions 2.2 and 2.4, we obtain the criterion announced in the Introduction.

THEOREM 2.6. *Let $F\colon \mathbf{R}^n \to \mathbf{R}^n$ be a polynomial mapping such that* $\deg F \geq 3$, $\operatorname{Jac} F \neq 0$ *in* $\mathbf{R}[x_1, \dots, x_n]$, *and F_{red} satisfies condition* $(\mathrm{J_r})$. *Then the mapping F is a polynomial-rational bijection of* $\mathbf{R}^n$.

To ensure that the class of polynomial transformations satisfying condition $(\mathrm{J_r})$ is nontrivial we propose

EXAMPLE 2.7. Let $F\colon \mathbf{R}^3 \ni (x_1, x_2, x_3) \to (x_1, x_2 - x_1^2x_3, x_3 + x_1^2x_2) \in \mathbf{R}^3$. Then $F = I + H$, where $H(x_1, x_2, x_3) = (0, -x_1^2x_3, x_1^2x_2)$ and, for every $x, y \in \mathbf{R}^n$, we have

$$\det(I + \hat{H}(x, x, \cdot) + \hat{H}(y, y, \cdot)) = 1 + \tfrac{1}{9}(x_1 + y_1)^2,$$
$$\det(I + \hat{H}(x, x, \cdot) - \hat{H}(y, y, \cdot)) = 1 + \tfrac{1}{9}(x_1 - y_1)^2.$$

Therefore, F satisfies condition $(\mathrm{J_r})$ and hence it is a polynomial-rational bijection of $\mathbf{R}^3$. An explicit formula for the inverse is

$$F^{-1}(x_1, x_2, x_3) = (x_1, (x_2 + x_1^2x_3)/(1 + x_1^4), (x_3 - x_1^2x_2)/(1 + x_1^4)), \quad (x_1, x_2, x_3) \in \mathbf{R}^3.$$

3. Polynomial transformations of degree two. In this case the real Jacobian Conjecture can be easily proved. Simultaneously, we obtain a complete characterization of polynomial-rational bijections of degree two.

THEOREM 3.1. *Let $F\colon \mathbf{R}^n \to \mathbf{R}^n$ be a polynomial mapping of degree two. Then the following statements are equivalent:*

(i) $\operatorname{Jac} F$ *has no zeros on* $\mathbf{R}^n$.

(ii) *F is injective.*

(iii) *F is a polynomial-rational bijection of* $\mathbf{R}^n$.

PROOF. The equivalence (i)$\Leftrightarrow$(ii) is straightforward. Indeed, if $\operatorname{Jac} F \neq 0$ in $\mathbf{R}^n$, then the injectivity of F follows immediately from the following obvious "mean-value theorem":

$$(*) \qquad F(x) - F(y) = F'((x + y)/2)(x - y) \quad \text{for every } x, y \in \mathbf{R}^n.$$

Conversely, assume that F is injective and suppose that $F'(u)(w) = 0$ for some points $u, w \in \mathbf{R}^n$, $w \neq 0$. Then, by $(*)$, we obtain the equality $F(u + w/2) = F(u - w/2)$, which contradicts the injectivity of F.

(ii)$\Rightarrow$(iii). Without loss of generality one can assume that $F = I + H$, where H is a homogeneous polynomial of degree two.

Then $\tilde{F}(x + iy) = x + H(x) - H(y) + iF'(x)(y)$ for every $x, y \in \mathbf{R}^n$. Suppose that $\tilde{F}(x + iy) \in \mathbf{R}^n$ for some $x, y \in \mathbf{R}^n$, i.e. $F'(x)(y) = 0$. According to (i) (which is equivalent to (ii)), the linear mapping $F'(x)$ is an isomorphism, so $y = 0$.

Therefore, for our injective mapping F we have also the equality $\tilde{F}^{-1}(\mathbf{R}^n) = \mathbf{R}^n$. Applying the surjectivity theorem [**2**] and Proposition 2.1, we obtain (iii).

The implication (iii)$\Rightarrow$(ii) is trivial. Q.E.D.

REMARK 3.2. It was first proved by Wang [**8**, Theorem 62] that a polynomial mapping F: $K^n \to K^n$ of degree two (where K denotes an arbitrary field of characteristic $\neq 2$) is a polynomial bijection of K^n with a polynomial inverse if and only if $\operatorname{Jac} F$ is a nonzero constant. In the real or complex case, this theorem follows relatively easily from the mean-value formula $(*)$ and standard facts from complex analysis.

Finally let us give an example of a polynomial transformation of degree two with nonconstant Jacobian without zeros on $\mathbf{R}^n$.

EXAMPLE 3.3. Let F: $\mathbf{R}^3 \ni (x_1, x_2, x_3) \to (x_1, x_2 - x_1x_3, x_3 + x_1x_2) \in \mathbf{R}^3$. Then $\operatorname{Jac} F = 1 + x_1^2$, so F is a polynomial-rational bijection of $\mathbf{R}^3$. An explicit formula for the inverse is

$$F^{-1}(x_1, x_2, x_3) = (x_1, (x_2+x_1x_3)/(1+x_1^2), (x_3-x_1x_2)/(1+x_1^2)), \quad (x_1, x_2, x_3) \in \mathbf{R}^n.$$

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