PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 102, Number 4, April 1988

A NOTE ON THE CUP PRODUCT FOR PRO-p GROUPS

TILMANN WÜRFEL

(Communicated by Bhama Srinivasan)

ABSTRACT. Let G be a pro-p group and $g \in H^1(G)$. We give a group-theoretic description of the kernel of the cup product $-\cup g\colon H^1(G) \to H^2(G)$.

In this note we exhibit a connection between the cup product $H^1(G) \times H^1(G) \xrightarrow{\cup} H^2(G)$ for a pro-p group G and certain subgroups of its Frattini group G^*. A typical result is the following (Corollary 1): If S and T are two different maximal subgroups of G, then $H^1(G/S) \cup H^1(G/T) = 0$ in $H^2(G)$ if and only if $S^*T^* \neq G^*$. We actually compute the annihilator of $H^1(G/S)$, with respect to the cup product, as $H^1(G/\hat{S})$ where $\hat{S}$ is determined by certain commutator conditions (Proposition 1).

We use standard notations. The basic facts about pro-p groups and their cohomology used here can be found in [**1**]. In particular, p always denotes a prime, G_i is the descending central series of a pro-p group G, $G^* = G^pG_2$, $H^i(G) = H^i(G, \mathbb{Z}/(p))$, $[x,y] = x^{-1}y^{-1}xy$, and $\langle \cdots \rangle$ denotes closed subgroups.

PROPOSITION 1. *Let G be a pro-p group and S a maximal subgroup of G. Fix an element $g \in H^1(G)$ such that $S = \operatorname{Ker}(g)$ and let $A_S = \{f \in H^1(G) \mid f \cup g = 0\}$. Denote by $\tilde{S}$ the subgroup $S^*\langle G^p\rangle G_3$ if p is odd, or S^*G_3 if $p = 2$. Then $A_S = H^1(G/\hat{S})$, where $\hat{S} = \{s \in S \mid [G,s] \subset \tilde{S}\}$ if p is odd or if $p = 2$ and $S^*G_2 \neq G^*$, and $\hat{S} = \{x \in G \mid [G,x] \subset \tilde{S}\}$ if $p = 2$ and $S^*G_2 = G^*$.*

The proof will follow from Proposition 2. First, we need some information about certain maximal subgroups of the Frattini group.

LEMMA. *Let G be a pro-p group and $S \leq G$ a maximal subgroup.*

(a) *If $T \leq G$ is a maximal subgroup different from S, then S^*T^* is maximal in or equal to G^* and contains G_3. If p is odd, then $G^p \subset S^*T^*$. If $p = 2$ and $S^*T^* \neq G^*$, then $G_2 \nleq S^*T^*$.*

(b) *Let p be odd. If $W \leq G^*$ is a maximal subgroup containing $S^*\langle G^p\rangle G_3$, then there exists a maximal $T \leq G$ such that $T \neq S$ and $W = S^*T^*$.*

(c) *Let $p = 2$. Every maximal subgroup $W \leq G^*$ such that $S^*G_3 \leq W$ and $G_2 \nleq W$ is of the form $W = S^*T^*$ with some maximal $T \leq G$ different from S.*

(d) *S^*G_2 is maximal in or equal to G^*; if $p = 2$, then this depends on whether or not $g \cup g$ is zero where $g \in H^1(G)$ with $\operatorname{Ker}(g) = S$.*

Received by the editors March 26, 1986 and, in revised form, December 18, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 20E18, 20J05.

PROOF. Everything follows from the congruence (cf. [**1**, Proposition 5])

$$(st)^p \equiv s^p t^p [t,s]^{\binom{p}{2}} \mod (G^*)^p[G^*,G], \tag{1}$$

where $[G^*,G] = (G_2)^p G_3$.

(a) To see that $G_3 \leq S^*T^*$, let $\overline{G} = G/S^*T^*$. Since $G = ST$, we can write $\overline{G} = \overline{S}\,\overline{T}$ with $\overline{S}, \overline{T}$ abelian and normal in $\overline{G}$. So $\overline{G}_2 = [\overline{S}, \overline{T}] \leq \overline{S} \cap \overline{T}$ which is contained in the center of $\overline{G}$. Hence $\overline{G}_3 = 1$.

Now pick an element $t \in T \backslash S$ and define the map $k\colon G \to G^*/S^*T^*$ by $k(x) = \overline{[t,x]}$. Since $G_3 \leq S^*T^*$, k is homomorphism with $T \leq \mathrm{Ker}(k)$. If p is odd, then (1) yields $G^p \subset S^*T^*$. Since $G = \langle t \rangle S$, we can write $G_2 = [G,S] = [t,S]S_2G_3$ which implies that k is surjective. So S^*T^* is maximal in or equal to G^* if p is odd. If $p = 2$, surjectivity of k follows from the congruence $(t^a s)^2 \equiv t^{2a}s^2[s,t^a] \equiv [t, s^{-a}] \mod S^*T^*$. Still in the case $p = 2$, assume that $G_2 \leq S^*T^*$. Then the congruence $(st)^2 \equiv s^2t^2[s,t] \equiv 1 \mod S^*T^*$ shows that $S^*T^* = G^*$.

(b) Since $G^p \subset W$, $G_2 = [G,S] \not\subset W$ and so there is a $t \in G$ with $[t,S] \not\subset W$. Then $t \notin S$ and the map $k\colon G \to G^*/W$ with $k(x) = \overline{[t,x]}$ is an epimorphism as in part (a) of the proof. The subgroup $T = \mathrm{Ker}(k)$ is maximal in G and different from S. Using $T = \langle t \rangle (T \cap S)$ we deduce that $T_2 \leq W$, so $S^*T^* = W$ by (a).

(c) As in the proof of (b) there is an element $u \in G$ so that $[u,S] \not\subset W$. The homomorphism $k\colon G \to G^*/W$, defined by $k(x) = \overline{[u,x]}$, is then surjective on S already. So there is an element $s \in S$ with $[u,s] \equiv u^{-2} \mod W$. Let $t = su$. We then have $t^2 \equiv s^2u^2[u,s] \equiv 1 \mod W$ by (1) and because $G_3 \leq W$. The modifed homomorphism $\tilde{k}\colon G \to G^*/W$ with $\tilde{k}(x) = \overline{[t,x]}$ now furnishes the required maximal subgroup $T = \mathrm{Ker}(\tilde{k})$ of G. Indeed, for $y \in S$ we have $[t,y] = [s,y]\,[[s,y],u]\,[u,y] \equiv [u,y] \mod S_2$, hence $\tilde{k}$ and k coincide on S, $t \notin S$, and $\tilde{k}$ is surjective. It remains to verify that $T^2 \subset W$. Write $v \in T$ as $v = t^a y$ with $y \in T \cap S$. Then $v^2 \equiv t^{2a}y^2[y,t^a] \equiv [t, y^{-a}] \equiv 1 \mod W$.

(d) The assignment $x \mapsto x^p$ induces an epimorphism $G \to G^*/S^*G_2$ whose kernel contains S. To prove the second statement, let $p = 2$ and consider the five term cohomology sequence associated with the group extension $1 \to S \to G \to \overline{G} \to 1$ where $\overline{G} = G/S$, together with the respective cup product homomorphisms $\cup_G$ and $\cup_{\overline{G}}$:

$$\begin{array}{ccccccc}
H^1(G) & \xrightarrow{r} & H^1(S)^{\overline{G}} & \longrightarrow & H^2(\overline{G}) & \longrightarrow & H^2(G) \\
 & & & & \uparrow{\scriptstyle \cup_{\overline{G}}} & & \uparrow{\scriptstyle \cup_G} \\
 & & & & H^1(\overline{G})^{\otimes 2} & \longrightarrow & H^1(G)^{\otimes 2}
\end{array}$$

Since $H^2(\overline{G}) = \mathbb{Z}/(2)$ and $\cup_{\overline{G}}$ is an isomorphism, we see that $g \cup_G g = 0$ if and only if the restriction r is not surjective which in turn is equivalent to $G^* \neq S^*G_2$. □

PROPOSITION 2. *Using the notation of Proposition* 1, *there is an exact sequence induced by the differential*

$$0 \to K_S \to A_S \xrightarrow{d} H^1(G^*/\tilde{S}) \to 0,$$

where $K_S = H^1(G/S)$ *if* p *is odd, and* $K_S = 0$ *if* $p = 2$.

PROOF. We begin by constructing the homomorphism $d\colon A_S \to H^1(G^*)$. If $f \in A_S$, then there is a continuous map $d_f\colon G \to \mathbb{Z}/(p)$ such that

$$f(x)g(y) = d_f(x) + d_f(y) - d_f(xy) \quad \text{for all } x, y \in G. \tag{2}$$

It follows that d_f is multiplicative on S as well as on $\mathrm{Ker}(f)$. So , in particular, the restricted map $d_f\,|_{G^*}$ lies in $H^1(G^*)$. If $\tilde{d}_f$ is another map satisfying (2), then $d_f - \tilde{d}_f$ is in $H^1(G)$ and hence vanishes on G^*. We can thus define the map d by setting $d(f) = d_f\,|_{G^*}$. By the linearity of (2), d is a homomorphism.

We show next that $\mathrm{Ker}(d) \subset H^1(G/S)$. Assume $d(f) = 0$, i.e., $d_f(G^*) = 0$. Since $d_f\,|_S \in H^1(S)$, there is then a map $c \in H^1(G)$ such that $c\,|_S = d_f\,|_S$. The modified map $\tilde{d}_f = d_f - c$ satisfies (2) and vanishes on S. So we may assume that $d_f(S) = 0$. We want to show that $f(S) = 0$. To this end, pick some $s \in S$ and let $z \in G$ be such that $g(z) = 1 \in \mathbb{Z}/(p)$. Then, by (2), $f(s) = d_f(z) - d_f(sz)$ and also $d_f(sz) = d_f(zs^z) = d_f(z)$. Hence $f(s) = 0$.

If p is odd, then the formula $g(x)g(y) = \frac{1}{2}(g(xy)^2 - g(x)^2 - g(y)^2)$ shows that $d(g) = 0$. Hence $\mathrm{Ker}(d) = H^1(G/S)$ in this case.

Let $p = 2$. If $g \cup g \neq 0$, then $H^1(G/S) \cap A_S = 0$ and so $\mathrm{Ker}(d) = 0$ by the above. If $g \cup g = 0$, then (2) implies that $g(x) = g(x)^2 = d_g(x^2)$ for all $x \in G$. But $G^* = \langle G^2 \rangle$ in this case (by (1)), hence $d(g) \neq 0$ and $\mathrm{Ker}(d) = 0$.

To see that d actually lands in $H^1(G^*/\widetilde{S})$, let $f \in A_S \backslash K_S$ and put $T = \mathrm{Ker}(f)$. Then $d_f(S^*T^*) = 0$. If $T \neq S$, then $\widetilde{S} \leq S^*T^*$ by (a) of the Lemma. If $T = S$, then $p = 2$ and we may assume $f = g$. Thus $g \cup g = 0$ and (2) implies that $d_g(xy) = d_f(yx)$ for all $x,\ y \in G$. Since $d_g\,|_S$ is a homomorphism, it follows that $d_g([s,x]) = d_g(s^{-1}) + d_g(s^x) = 0$ for all $s \in S,\ x \in G$. Hence $d_g(G_2) = 0$ and, in particular,

$$d_g(\widetilde{S}) = 0. \tag{3}$$

To show that $d(A_S) = H^1(G^*/\widetilde{S})$, we deal with odd p first. Let $0 \neq \varphi \in H^1(G^*)$ be such that $\varphi(\widetilde{S}) = 0$. We want to produce an $f \in H^1(G)$ such that $f \cup g = 0$ and $d(f) = \varphi$. Let $z \in G$ be such that $g(z) = 1$ and define f by $f(x) = \varphi([z,x])$. Then $f \in H^1(G)$ because $G_3 \leq \widetilde{S}$. Let $\theta = \sigma\pi$ where $\pi\colon G \to G/S$ is the natural map and $\sigma\colon G/S = \langle \bar{z} \rangle \to G$ is the section defined by $\sigma(\bar{z}^i) = z^i$ for $i = 1, \ldots, p-1$. Since $S/\widetilde{S}$ is p-elementary, we may assume that φ is actually in $H^1(S)$, and can thus define a continuous map $c\colon G \to \mathbb{Z}/(p)$ by setting $c(x) = \varphi(\theta(x)^{-1}x)$. Then $c(xs) = c(x) + \varphi(s)$ for all $x \in G,\ s \in S$. So $c\,|_{G^*} = \varphi$. We want to show that $f(x)g(y) = c(x) + c(y) - c(xy)$ for all $x,\ y \in G$. Write $x = z^i r$ and $y = z^j s$ where $0 \leq i,j \leq p-1$ and $r, s \in S$. Then $f(x)g(y) = jf(x)$ and

$$c(x) + c(y) - c(xy) = \varphi(r) + \varphi(s) - c(z^i r z^j) - \varphi(s) = \varphi(r) - c(z^i r z^j).$$

Write $i + j = ap + m$ with $0 \leq m \leq p-1$. Then $z^i r z^j = z^m z^{ap} r [r, z^j]$ and, since $\varphi(z^p) = 0$, $c(z^i r z^j) = \varphi(r) + \varphi([r, z^j]) = \varphi(r) - jf(r)$ which is what we need because $f(r) = f(x)$.

To end the proof, let $p = 2$ and let φ be as above. If $G_2 \nleq \mathrm{Ker}(\varphi)$, then, by (c) of the Lemma, $\mathrm{Ker}(\varphi) = T^*S^*$ with some maximal $T \leq G$ different from S. So there is an element $z \in T$ such that $g(z) = 1$. Then $\varphi(z^2) = 0$ and we can proceed as above. If $G_2 \leq \mathrm{Ker}(\varphi)$, then $\mathrm{Ker}(\varphi) = S^*G_2$ and $g \cup g = 0$ by (d) of the Lemma. In this case, by (3), we have $d_g(\widetilde{S}) = 0$. Since $d_g(G^*) \neq 0$, it follows that $d_g\,|_{G^*} = \varphi$. □

COROLLARY 1. *Let G be a pro-p group and $f, g \in H^1(G)$ be linearly independent with $T = \mathrm{Ker}(f)$, $S = \mathrm{Ker}(g)$. Then $f \cup g = 0$ if and only if $S^*T^* \neq G^*$. Hence $f \cup g \neq 0$ if G is abelian.*

PROOF. $S \neq T$ by assumption. If $f \cup g = 0$, then, by Proposition 2, $\varphi = d(f) \neq 0$. Since $\varphi(S^*T^*) = 0$, we have that $S^*T^* \neq G^*$. Conversely, if this latter condition holds, then there is a nonzero $\varphi \in H^1(G^*)$ with $\varphi(S^*T^*) = 0$. If $p = 2$, then $G_2 \nleq \mathrm{Ker}(\varphi)$ by (a) of the Lemma. Define $\tilde{f} \in H^1(G)$ by $\tilde{f}(x) = \varphi([z, x])$ where $z \in T$ is such that $g(z) = 1$. Then $\tilde{f}(T) = 0$, so $\tilde{f} = af$ with some $a \in \mathbb{Z}/(p)$. By the proof of Proposition 2, we have $\tilde{f} \cup g = 0$ and $d(\tilde{f}) = \varphi \neq 0$. Hence $a \neq 0$ and $f \cup g = 0$. □

COROLLARY 2. *Let G be a pro-p group of finite rank with minimal pro-p free presentation $1 \to R \to F \xrightarrow{\pi} G \to 0$. Let $S \leq G$ be a maximal subgroup and put $U = \pi^{-1}(S)$. Then $H^1(G/S)$ is contained in the radical of the cup product $H^1(G) \times H^1(G) \to H^2(G)$ if and only if $R \leq U^*\langle F^p\rangle F_3$ if p is odd, or $R \leq U^*F_3$ if $p = 2$.*

PROOF. Define $\widetilde{S}$ and $\widetilde{U}$ as in Proposition 1. Then π induces an epimorphism $\bar{\pi}\colon F^*/\widetilde{U} \to G^*/\widetilde{S}$ whose kernel is $R\widetilde{U}/\widetilde{U}$. Therefore, $R \leq \widetilde{U}$ if and only if the groups $F^*/\widetilde{U}$ and $G^*/\widetilde{S}$ are of the same rank. Let n denote the rank of G and F. Since $H^2(F) = 0$, Proposition 2 shows that the rank of $F^*/\widetilde{U}$ is $n - 1$ if p is odd and n if $p = 2$. On the other hand, $H^1(G/S)$ is contained in the radical of the cup product for G if and only if $A_S = H^1(G)$ which, by Proposition 2, is equivalent to $G^*/\widetilde{S}$ having rank $n - 1$ if p is odd and rank n if $p = 2$. □

PROOF OF PROPOSITION 1. We refer to the proof of Proposition 2. Let $z \in G$ be such that $g(z) = 1$. The map $s \mapsto [z, s]$ then induces a homomorphism $k\colon S/G^* \to G^*/\widetilde{S}$ which does not depend on the choice of z. By the second part of the proof of Proposition 2, the following diagram is commutative:

$$\begin{array}{ccc} A_s & \xrightarrow{\ d\ } & H^1(G^*/\widetilde{S}) \\ \big\downarrow & & \big\downarrow{\scriptstyle \bar{k}} \\ H^1(G) & \xrightarrow{\ r\ } & H^1(S/G^*) \end{array}$$

where $\bar{k}$ is the dual of k and r is restriction. The snake lemma applied to this diagram enlarged by the appropriate kernels and cokernels now yields that $A_s = H^1(G/\hat{S})$ if p is odd, or if $p = 2$ and $S^*G_2 \neq G^*$, hence $H^1(G^*/S^*G_2) = H^1(G/S)$.

If $p = 2$ and $S^*G_2 = G^*$, then, in a way similar to the proof of (c) of the Lemma, an element $z \in G$ such that $g(z) = 1$ can be found which satisfies $z^2 \in \widetilde{S}$. Let $k\colon G \to G^*/\widetilde{S}$ be the homomorphism induced by $x \mapsto [z, x]$. Then the proof of Proposition 2 shows that

$$\begin{array}{ccc} A_S & \xrightarrow{\ d\ } & H^1(G^*/\widetilde{S}) \\ \big\downarrow & \swarrow{\scriptstyle \bar{k}} & \\ H^1(G) & & \end{array}$$

is commutative. Hence $A_S = H^1(G/\hat{S})$ in this case, too. □

REMARK. Let p be an odd prime and G a pro-p group with a maximal subgroup S. By Proposition 2, the annihilator A_S, with respect to the cup product, of $H^1(G/S)$ in $H^1(G)$ is smallest possible, i.e., $A_S = H^1(G/S)$, exactly when $G^* = S_2\langle G^p\rangle G_3$. Let (C) denote this condition if it holds for all maximal $S \leq G$. It is obviously satisfied if $G_2 \leq \langle G^p\rangle$ in which case G is called powerful in [**2**]. The converse holds if the rank n_G of G is ≤ 3, but fails for $n_G = 4$.

PROOF. (i) Let G satisfy (C) and $n_G \leq 3$. We can assume that $G^p = 1$ and $G_3 = 1$ because G satisfies (C) if and only if $G/\langle G^p\rangle G_3$ does, and the same is true for powerfulness. So it remains to show that G is abelian. We have $S_2 = G_2$ for all maximal $S \leq G$. This settles the case $n_G = 2$ because $S/S_2 \hookrightarrow G/G_2$ implies that $n_S = n_G - 1 = 1$, hence $S_2 = 1$. Now let $n_G = 3$, $G = \langle x_1, x_2, x_3\rangle$, and assume that $G_2 \neq 1$. Then $G_2 = \mathbb{Z}/(p)$ by using some maximal subgroup as above. Let $G_2 = \langle [x_1, x_2]\rangle$, say. Then $[x_3, x_1] = [x_2, x_1]^a = [x_2^a, x_1]$ for some $a \in \mathbb{Z}$. Let $\tilde{x}_3 = x_3 x_2^{-a}$. Then $[\tilde{x}_3, x_1] = 1$ and $G = \langle x_1, x_2, \tilde{x}_3\rangle$. The subgroup $S = \langle x_1, \tilde{x}_3\rangle G_2$ is maximal because $G = \langle x_2\rangle S$, but $S_2 = 1$.

(ii) Let $G = \langle x_1, x_2, x_3, x_4\rangle$ satisfy the relations $G^p = 1$, $G_3 = 1$, $[\langle x_1, x_2\rangle, \langle x_3, x_4\rangle] = 1$, and $[x_1, x_2] = [x_3, x_4]$ $(\neq 1)$. Then G is not powerful. Since $G_2 = \mathbb{Z}/(p)$, condition (C) just means that $S_2 \neq 1$ for all maximal $S \leq G$. Assume there is an S with $S_2 = 1$. Then $S = C_G(s)$, the centralizer of any $s \in S$ which is not in the center $Z(G)$. We have that $x_1 \notin S$ because otherwise, since $x_1 \notin Z(G)$, x_3 and x_4 would be in S and would hence commute. Therefore, $G = \langle x_1\rangle S$ and $x_2 = x_1^a s$ for some $a \in \mathbb{Z}$, $s \in S$. Then $s = x_1^{-a}x_2 \notin Z(G)$ because $[s, x_1] = [x_2, x_1]$. But $x_3, x_4 \in C_G(s) = S$ which is impossible. □

ADDED IN PROOF. After submitting this note we became aware that there is an overlap with two papers by J. A. Hillman, *The kernel of the cup product*, Bull. Austral. Math. Soc. **32** (1985), 261–274 and J. Austral. Math. Soc. Ser. A **43** (1987), 10–15. The paper by Lubotzky and Mann has appeared in J. Algebra **105** (1987), 484–515.

REFERENCES

1. J. P. Labute, *Demuškin groups of rank* $\aleph_0$, Bull. Soc. Math. France **94** (1966), 211–244.
2. A. Lubotzky and A. Mann, *Powerful p-groups*, preprint.

DEPARTMENT OF MATHEMATICS, PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PENNSYLVANIA 16802

Current address: Pennsylvania State University, Delware County Campus, Media, Pennsylvania 19063