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ON EXPONENTS OF HOMOLOGY AND COHOMOLOGY OF FINITE GROUPS

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ABSTRACT. Let G be a finite group and let r be the maximum of the p-ranks of G for all primes p dividing the order to G. There exist positive integers m and n such that the exponents of $H^n(G,\mathbf{Z})$ and $H_m(G,\mathbf{Z})$ are greater than $|G|^{1/r}$.

Suppose that G is a finite group. It has long been known that the exponent of $H^n(G,\mathbf{Z})$ divides $|G|$, the order of G, for all $n > 0$. Beyond this fact very little has been established concerning exponents of cohomology groups, and what is known seems to be mostly about upper bounds for the exponents. Lewis **[3]** found an element of order p^2 in $H^{2p}(G,\mathbf{Z})$ when G is the group of order p^3 and exponent p (p odd). The purpose of this paper is to give a general lower bound for exponents on group homology and cohomology. In some sense it explains the phenomenon observed by Lewis.

Throughout let R denote either the ordinary integers $\mathbf{Z}$ or the localization $\mathbf{Z}_p$ of $\mathbf{Z}$ at a prime p. Define the R-rank of G to be equal to the p-rank of G if $R = \mathbf{Z}_p$ and equal to the maximum of the p-ranks of G for all primes p if $R = \mathbf{Z}$. The main result is the following.

THEOREM 1. *Let r be the R-rank of G. There exist an infinite number of positive integers m and n such that*

$$\exp H^n(G,R) \geq |R/(g)|^{1/r} \quad \textit{and} \quad \exp H^m(G,R) \geq |R/(g)|^{1/r},$$

where $|R/(g)|$ is the order of the group $R/(g)$, $g = |G|$.

In the case $R = \mathbf{Z}$, $|R/(g)| = |G|$. If $R = \mathbf{Z}_p$ and $|G| = p^a \cdot q$, then $R/(g)$ has order p^a. It is well known that, because $\mathbf{Z}_p$ is a flat $\mathbf{Z}$-module, $H^n(G,\mathbf{Z}) = \bigoplus_p H^n(G,\mathbf{Z}_p)$ and $H_m(G,\mathbf{Z}) = \bigoplus_p H_m(G,\mathbf{Z}_p)$ for $m,n \geq 1$. That is, $H^n(G,\mathbf{Z}_p)$ is isomorphic to $H^n(G,\mathbf{Z})_p$, the p-primary part of $H^n(G,\mathbf{Z})$. Hence in case $R = \mathbf{Z}_p$ the theorem implies the following.

COROLLARY 2. *Suppose that $|G| = p^a \cdot q$, $(p,q) = 1$, and $a \geq 1$. Let r be the p-rank of G, and let s be the least integer that is greater than or equal to a/r. Then there exist an infinite number of positive integers m and n such that p^s divides $\exp H^n(G,\mathbf{Z})$ and $\exp H_m(G,\mathbf{Z})$.*

The proof of Theorem 1 is little more than a combination of two results. The first demonstrates that certain types of complexes of projective RG-modules can

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be constructed [**1**]. The theorem is then almost immediate from the second result [**2**]. For the sake of completeness we present a brief description of the construction. (See [**1**] for details and references.)

Let (p) be a maximal prime ideal in R. For $\varsigma \in H^n(G,R)$, let $\bar{\varsigma} \in H^n(G,R/(p))$ denote its reduction modulo p. Let r be the R-rank of G and choose homogeneous elements $\varsigma_1,\dots,\varsigma_r \in H^*(G,R)$ such that, for every such prime ideal (p), the elements $\bar{\varsigma}_1,\dots,\bar{\varsigma}_r$ have the property that $H^*(G,R/(p))/(\bar{\varsigma}_1,\dots,\bar{\varsigma}_r)$ is a finite-dimensional algebra over $R/(p)$. Also if $R=\mathbf{Z}$ and r_p is the p-rank of G, then assume that $(\bar{\varsigma}_1,\dots,\bar{\varsigma}_{r_p})$ generate an ideal of finite codimension in $H^*(G,R/(p))$. If $V_G(R/(p))$ is the maximal ideal spectrum of $H^*(G,R/(p))/\operatorname{Rad} H^*(G,R/(p))$ and if $V_G(\bar{\varsigma}_i)$ is the subvariety of $V_G(R/(p))$ corresponding to the ideal generated by $\bar{\varsigma}_i$, the preceding is equivalent to the statement that

$$V_G(\bar{\varsigma}_1)\cap\cdots\cap V_G(\bar{\varsigma}_r)=\{0\}.$$

Such sets of elements exist because r_p is the Krull dimension of $H^*(G,R/(p))$.

Let

$$\cdots \to F_1 \to F_0 \to R \to 0$$

be an RG-projective resolution of R. We need two more assumptions, neither of which cause any problems as far as the existence of sets $\{\varsigma_1,\dots,\varsigma_r\}$ as above. First assume that $l_i=\deg(\varsigma_i)>1$. Also if $R=\mathbf{Z}$ we insist that the p-primary part of ς_i is zero if $i>r_p$, the p-rank of G. Then each ς_i is represented by a cocycle $\hat{\varsigma}_i\colon F_{l_i}\to R$ which is a surjection. Now consider the following commutative diagram:

$$\begin{array}{ccccccccccccccccc}
\cdots & \to & F_{l_i+1} & \to & F_{l_i} & \to & F_{l_i-1} & \to & F_{l_i-2} & \to & \cdots & \to & F_0 & \to & R & \to & 0\\
 & & \downarrow & & \hat{\varsigma}_i\downarrow & & \downarrow & & \| & & & & \| & & \| & & \\
 & & 0 & \to & R & \to & L_i & \to & F_{l_i-2} & \to & \cdots & \to & F_0 & \to & R & \to & 0\\
 & & & & & & \| & & \| & & & & \| & & & & \\
 & & & & 0 & \to & C^{(i)}_{l_i-1} & \to & C^{(i)}_{l_i-2} & \to & \cdots & \to & C^{(i)}_0 & \to & 0 & &
\end{array}$$

Here the middle row is the pushout of the top row along $\hat{\varsigma}_i$. The bottom row is a complex which we denote by $C^{(i)}$. It is clear that $H_j(C^{(i)})=0=H^j(C^{(i)})$ if $j\neq 0$, l_i-1, while $\overline{H}_j(C^{(i)})\cong R\cong H^j(C^{(i)})$ if $j=0$ or $j=l_i-1$. From the conditions on $\varsigma_1,\dots,\varsigma_r$ it follows that $L=L_1\otimes\cdots\otimes L_r$ is a projective RG-module because $L/(p)L$ is a projective $(R/(p))G$-module for all primes p. Now let $C=C(\varsigma_1,\dots,\varsigma_r)$ be the total complex of the tensor product $C=C^{(1)}\otimes_R\cdots\otimes_R C^{(r)}$. In summary, using the Künneth tensor formula, we get the following.

THEOREM 3 [**1**]. *Let r be the R-rank of G. There exist homogeneous elements $\varsigma_1,\dots,\varsigma_r\in H^*(G,R)$ satisfying the following conditions.*

(i) *$H^*(G,R/(p))/(\bar{\varsigma}_1,\dots,\bar{\varsigma}_r)$ is a finite-dimensional algebra over $R/(p)$ for all maximal ideals $(p)\subseteq R$.*

(ii) $\deg\varsigma_i=l_i>1$ *for all i.*

(iii) *If $R=\mathbf{Z}$, then the p-primary part of ς_i is zero for $i>r_p$, the p-rank of G.*

If $\varsigma_1,\dots,\varsigma_r$ satisfy the above conditions, then $C=C(\varsigma_1,\dots,\varsigma_r)$ is a complex of projective RG-modules such that every $H_j(C)$ and $H^j(C)$ is a direct sum of copies of R. In particular

$$H^j(C)=\sum H^{j_1}(C^{(1)})\otimes\cdots\otimes H^{j_r}(C^{(r)})$$

and

$$H_j(C) = \sum H_{j_1}(C^{(1)}) \otimes \cdots \otimes H_{j_r}(C^{(r)}),$$

the sum being over all r-tuples $(j_1, \ldots, j_r)$ such that $j_1 + \cdots + j_r = j$.

The other fact that we require is the following. Browder's proof for the case $R = \mathbf{Z}$ is trivially generalized to the other situations.

THEOREM 4 [**2**]. *Let C be a finite nonnegative chain complex of projective RG-modules such that $H_0(C) = R$. Then as elements of R*

$$|G| \textit{ divides } \prod_{j \geq 1} \exp H^{j+1}(G, H_j(C)) \tag{1}$$

and

$$|G| \textit{ divides } \prod_{j \geq 1} \exp H_{j+1}(G, H^j(C)). \tag{2}$$

PROOF OF THEOREM 1. Suppose that r is the R-rank of G, and that $\varsigma_1, \ldots, \varsigma_r$ satisfy all of the conditions necessary for Theorem 3 to hold. It may be observed that $\varsigma_1^{n_1}, \ldots, \varsigma_r^{n_r}$ also satisfies the same hypotheses for any positive integers $n_1, \ldots, n_r$. Also $\deg(\varsigma_i^{n_i}) = n_i \deg(\varsigma_i)$, so we may assume that $\varsigma_1, \ldots, \varsigma_r$ all have the same degree l. Then the complex C has homology and cohomology only in degrees $j(l-1)$ for $j = 0, 1, \ldots, r$. Consequently Theorem 4 implies that $|G|$ divides $\prod_{j=1}^{r} \exp H^{j(l-1)+1}(G, R)$ in R. Thus $\exp H^n(G, R) \geq |R/(g)|^{1/r}$ for some j, where $n = j(l-1) + 1$. This proves existence of one such n. To obtain another simply replace every ς_i by ς_i^t for $t \cdot \deg \varsigma_i > n$ and repeat the process. The proof for homology is almost the same.

For cohomology the existence of one value of n automatically implies the existence of an infinite number, because the ring $H^*(G, R)$ has elements of positive degree that are not divisors of zero.

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