ON THE JACOBSON RADICAL OF SOME ENDOMORPHISM RINGS

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ABSTRACT. In this note we deal with a question raised by R. S. Pierce in 1963: Determine the elements of the Jacobson radical of the endomorphism ring of a primary abelian group by their action on the group. We concentrate on separable abelian p-groups and give a counterexample to a conjecture of A. D. Sands. We also show that the radical can be pinned down if the endomorphism ring is a split-extension of its ideal of all small maps.

Introduction. All groups in this note are abelian p-groups for some fixed but arbitrary prime p. Our notations are standard as in [F]. It is known that the endomorphism ring End(A) of an abelian p-group A determines the group up to isomorphism. R. Pierce [P] raised the question of describing the Jacobson radical J(End(A)) of End(A) by its action on the group. This problem was solved by W. Liebert [L], J. Hausen [H] and Hausen-Johnson [HJ] for Σ-cyclic, torsion-complete and sufficiently projective p-groups. (For a separable p-group sufficiently projective is the same as ω₁-separable.) If A is a (separable) p-group, let \( H(A) = \{ \varphi \in \text{End}(A) \mid |x| < |x\varphi| \text{ for all } 0 \neq x \in A[p] \} \) be the ideal of all maps acting height increasing on the socle of A, and let \( C(A) \) be the ideal of all elements of \( \text{End}(A) \) mapping each Cauchy sequence in \( A[p] \) onto a convergent one. (For \( x \in A \), \( |x| \) denotes the p-height of \( x \) in \( A \) and topological notations refer to the p-adic topology.) If A is torsion-complete, \( J(\text{End}(A)) = H(A) \), if A is Σ-cyclic or ω₁-separable, \( J(\text{End}(A)) = H(A) \cap C(A) \), and \( H(A) \cap C(A) \subset J(\text{End}(A)) \) for all separable p-groups (cf. [S]). The purpose of this paper is to show that \( J(\text{End}(A)) \) is in general not equal to \( C(A) \cap H(A) \) for separable p-groups A. We will use that \( J(\text{End}(A)) \cap E_s(A) = E_s(A) \cap H(A) \), where \( E_s(A) \) is the ideal of all small endomorphisms of \( A \) (cf. [S]). Recently, many complicated p-groups have been constructed in [DG1, DG2, CG]. All these groups enjoy the property that \( \text{End}(A) \) is a split extension of \( E_s(A) \), i.e. \( \text{End}(A) = R \oplus E_s(A) \) for some subring \( R \) of \( \text{End}(A) \). The way these groups are constructed, \( R \cap H(A) = pR \) and \( \overline{H}_R(A) = \overline{H}_R(A) \), i.e. if \( r \in R - H(A) \), then for all \( n \) there is \( 0 \neq x \in p^nA[p] \) such that \( x \) and \( xr \) have the same height. In this situation Theorem 1 below implies

\[
J(\text{End}(A)) = (J(R) \cap H(A)) \oplus (E_s(A) \cap H(A))
\]

and we have \( J(\text{End}(A)) = H(A) \cap C(A) \) for these groups. We will construct a ring \( R \) and use the realization result in [C] to obtain a separable p-group A such that

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End(A) = R ⊕ E_s(A) and (J(R) - pR) ∩ H(A) and (R - J(R)) ∩ H(A) are both not empty. Since C(A) ∩ R = pR for this group A, we have that H(A) ∩ C(A) is a proper subset of J(End(A)). This makes it hard to believe that Pierce's question has a positive answer for all separable p-groups.

**The construction.** Let A be a separable p-group and E = End(A) the endomorphism ring of A. If S ⊂ E, and U ⊂ A[p] are subsets, define \( \overline{H}_S(A, U) = \{ \varphi \in S \mid \exists x \in U, |x| = |x\varphi| \} \). Observe that \( \overline{H}_E(A, U) \cap S = \overline{H}_S(A, U) \) and if \( U \subset V \) we have \( \overline{H}_S(A, U) \subset \overline{H}_S(A, V) \). We also define \( H(A) = E - \overline{H}_E(A, A[p]) \), \( \overline{H}_S(A) = \bigcap_n \overline{H}_S(A, p^nA[p]) \) and \( \overline{H}_{S}(A) = S \cap H(A) \) and \( H_S(A, U) = S - \overline{H}_S(A, U) \).

**Theorem 1.** Let A be a separable p-group such that \( E = E(A) = R \oplus E_s(A) \) is a split extension of \( E_s(A) \) and \( \overline{H}_R(A) = \overline{H}_s^*(A) \). Then \( J(E) = H_{J(R)}(A) \oplus H_{E_s}(A) \).

**Proof.** Let \( r \in R \), and \( \sigma \in E_s(A) \) with \( r + \sigma \in J(E) \). Then \( (r + \sigma)t \) is right quasi-regular for all \( t \in R \) and, since \( E = R \oplus E_s(A) \), the element \( rt \) is right quasi-regular in \( R \) as well as \( r \in J(R) \). Now suppose \( r \notin H_{J(R)}(A) \). Since \( \sigma \) is small and \( \overline{H}_R(A) = \overline{H}_s^*(A) \), we find \( n < \omega \) and \( 0 \neq x \in p^nA[p] \) such that \( x\sigma = 0 \) and \( |x| = |x\sigma| \). Let \( F \) be a finite summand of \( A \) containing \( x\sigma \) and let \( \rho: A \to F \) be the natural projection. Then \( x\sigma = x\rho \) and \( |x\sigma| = |x\rho| = |x(r + \sigma)p| \). Since \( Ap = F \) is finite, \( \rho \in E_s(A) \) and hence \( (r + \sigma)\rho \in J(E) \cap E_s(A) = E_s(A) \cap H(A) \) (cf. [S]). This contradicts the above equation of heights and we conclude \( r \in H_{J(R)}(A) \). Now let \( t \in R \), \( \sigma \in E_s(A) \) and \( r \in H_{J(R)}(A) \). Then there is \( s \in R \) such that \( (1 - rt)s = 1 \). This implies \( (1 - (r + \sigma)s) = (1 - rt)s - \sigma s = 1 - \sigma s \). Since \( \sigma \in E_s(A) \) and \( r \in H_{J(R)}(A) \) we have that \( \sigma s \in H_{E_s}(A) \subset J(E) \) and there is \( r \in E \) with \( (1 - \sigma s)r = 1 \). This implies \( (1 - (r + \sigma))s = 1 \) and \( r \in J(E) \). We obtain \( H_{J(R)}(A) \subset J(E) \subset H_{J(R)}(A) \) which together with \( J(E) \cap E_s(A) = H_{E_s}(A) \) implies the desired equation.

We now construct our ring:

Let \( \omega \) be the set of natural numbers including 0 and let

\[
B = \bigoplus_{i \in \omega} (f_i \oplus g_i \oplus h_i)
\]

be a \( \Sigma \)-cyclic p-group with \( \exp(f_i) = i + 1 = \exp(g_i) \) and \( \exp(h_i) = i + 2 \). We define elements \( \alpha, \beta, \gamma \in \text{End}(B) \) by setting \( f_i \alpha = p f_{i+1}, f_i \beta = g_i \) and \( f_i \gamma = ph_i \), and \( \alpha, \beta \) and \( \gamma \) are 0 on the \( g_i \)'s and \( f_i \)'s. Let \( S = \{ 1, \alpha, \beta, \gamma \} \) be the subring of \( \text{End}(B) \) generated by these elements and \( R = \tilde{S} \) be the p-adic completion of \( S \). We have the following relations:

1. \( \beta \gamma = \gamma \alpha = \beta^2 = \gamma^2 = \beta \gamma = \gamma \beta = 0 \).
2. Each element \( r \in S \) has a unique representation:

\[
r = \sum_{i=0}^{n} \alpha^i a_i + \sum_{i=0}^{m} \alpha^i b_i + \sum_{i=0}^{k} \alpha^i c_i
\]

with \( a_i, b_i \) and \( c_i \) integers.

Therefore each element \( x \in R = \tilde{S} \) has a unique representation:

3. \( x = \sum_{i=0}^{\infty} \alpha^i a_i + \sum_{i=0}^{\infty} \alpha^i b_i + \sum_{i=0}^{\infty} \alpha^i c_i \) where \( \{ a_i \}, \{ b_i \} \) and \( \{ c_i \} \) are p-adic zero-sequences in \( J \), the ring of p-adic integers. Let \( I \) be the set of all \( x \in R \) with all \( a_i \)'s being 0. This is the ideal of \( R \) generated by \( \beta \) and \( \gamma \). An easy computation shows:
(4) Let $x \in R$ be as in (3). Then $\exp(f_kx) = \max_{i \in \omega}\{k + 1 - |a_i|, k + 1 - |b_i|, k + 1 - |c_i|\}$. Here the max is defined to be 0 if all numbers in the set are $< 0$.
Because of (3), $\{a^i \mid i < \omega\}$ is linearly independent and we have

(5) $R/I \cong \left(J_p[\alpha]\right)^{-},$ the p-adic completion of the polynomial ring $J_p[\alpha]$.
This implies

(6) $R/(pR + I) \cong GF(p)[\alpha],$ the polynomial ring over $GF(p)$.
This implies $pR \subset J(R) \subset pR + I$.

(7) Let $j < k + 1$. Then $|p^j f_kx| > j = |p^j f_k|$ for all $x \in I$, the ideal generated by $\beta$ and $\gamma$, i.e. $I \subset H(\overline{B})$.
This follows from a straightforward computation using that $B$ is $\Sigma$-cyclic. Observe that $I^2 = 0$.

(8) $J(R) = pR + I$.
We want $R$ to be pure in $E(\overline{B}), \overline{B}$ the torsion-completion of $B$. We show a little more:

(9) $R \oplus E_s(\overline{B})$ is pure in $E(\overline{B})$. (One needs this to do a “Black Box” construction; cf. [CG].)

To prove (9), let $\varphi \in E(\overline{B}), \tau \in R, \sigma \in E_s(\overline{B})$ and $n \in \omega$ with $p^n\varphi = r + \sigma$. Since $\sigma$ is small, there is $k \in \omega$ such that $p^{k-n-1}f_k p^n \varphi = p^{k-n-1} f_k r$ and $p^2 p^{k-1} f_k \varphi = 0$. Hence $p^{k-n+1} f_k r = 0$. Now let $\tau$ be represented as in (3) and apply (4) to obtain:

$$k - n + 1 > \exp(f_k \tau) = \max_{i \in \omega}\{k + 1 - |a_i|, k + 1 - |b_i|, k + 1 - |c_i|\}.$$ 

This implies $k - n + 1 \geq k + 1 - |a_i|$ and $|a_i| \geq n$. The same holds for the $b_i$'s and $c_i$'s. Therefore $\tau \in p^n R$ and $\tau = p^n s$ for some $s \in R$. Thus $p^n(\varphi - s) = \sigma$ is small which implies $\varphi - s$ is small and $\varphi \in R \oplus E_s(A)$. \(\square\)

(10) Let $A$ be a pure subgroup of $\overline{B}$ containing $B$ and $\varphi \in H_E(A)(A, B[p])$. Then $\varphi \in H_{E_s}(A)(A, A[p])$.

Let $a \in A[p] - B[p]$. Since $A/B$ is divisible, there is $b \in B, y \in A$ such that $a = b + p^{n+1} y$ where $n = |a|$. Then $|a| = |b|$ and $|a \varphi| = |(b + p^{n+1} y) \varphi| \geq n + 1 > n = |a|$ since $\varphi \in H_E(A)(A, B[p])$. This inequality shows $\varphi \in H_{E_s}(A)(A, A[p])$. \(\square\)

Now we apply A. L. S. Corner's result [C, Theorem 2.1] and obtain a pure subgroup $A$ of $\overline{B}$ containing $B$ and $\text{End}(A) = R \oplus E_s(A)$. (Observe that (4) implies that condition (C) of [C, Theorem 2.1] holds.) The ring $R$ is constructed to satisfy $H_R(A) = pR + \alpha R + R\gamma$. Moreover we have $\alpha \in h_R(A) - J(R), \beta \in J(R) - H_R(A)$ and $\gamma \in J_R(A) - pR$. Observe that $\gamma \not\in C(A)$, since otherwise $B[p] \gamma \subset A$.
In order to see that this is absurd, we have to look into Corner's proof [C] of his Theorem 2.1: Recall that for a positive integer $e$ an element $x \in \overline{B}$ is $e$-strong if $xr = 0$ implies $r \in p^e R$ for $x \in B[p]$. Moreover for $e = 1, x \in B[p]$ and since $\gamma \not\in pR$ we conclude $x\gamma \neq 0$. This shows that $B[p] \gamma$ is not contained in any $G_\sigma (= A)$. Now Theorem 1 applies and we have that $J(End(A)) = H_J(R)(A) \oplus H_{E_s}(A)(A)$ is not contained in $C(A) \cap H(A)$ since $\gamma$ is not and also $H_R(A)$ is not contained in $J(R)$. So if we want to describe the elements of $J(End(A))$ by their action on $A$, we have to find the elements in $J(R) \cap H(A)$, which means we must be able to recognize the elements of $J(R)$. There is much freedom for the way an element of $J(R)$ can operate on $A$. We answer a question...
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in [S] by summarizing part of our discussion in

THEOREM 2. There exists a separable $p$-group $A$ such that $J(\text{End}(A))$ is larger than $H(A) \cap C(A)$.

REMARK. If we want to have larger groups $A$ realizing $R$, we may employ Shelah's "Black Box" and a construction very similar to the one in [CG]. We would like to mention again that all the $p$-groups constructed in [CG, DG1 or DG2] satisfy $H(A) \cap J(R) = pR$.

REFERENCES


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