PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 102, Number 4, April 1988

ON THE JACOBSON RADICAL OF SOME ENDOMORPHISM RINGS

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(Communicated by Bhama Srinivasan)

ABSTRACT. In this note we deal with a question raised by R. S. Pierce in 1963: Determine the elements of the Jacobson radical of the endomorphism ring of a primary abelian group by their action on the group. We concentrate on separable abelian p-groups and give a counterexample to a conjecture of A. D. Sands. We also show that the radical can be pinned down if the endomorphism ring is a split-extension of its ideal of all small maps.

Introduction. All groups in this note are abelian p-groups for some fixed but arbitrary prime p. Our notations are standard as in [**F**]. It is known that the endomorphism ring $\text{End}(A)$ of an abelian p-group A determines the group up to isomorphism. R. Pierce [**P**] raised the question of describing the Jacobson radical $J(\text{End}(A))$ of $\text{End}(A)$ by its action on the group. This problem was solved by W. Liebert [**L**], J. Hausen [**H**] and Hausen-Johnson [**HJ**] for Σ-cyclic, torsion-complete and sufficiently projective p-groups. (For a separable p-group sufficiently projective is the same as ω_1-separable.) If A is a (separable) p-group, let $H(A) = \{\varphi \in \text{End}(A) |\ |x| < |x\varphi| \text{ for all } 0 \neq x \in A[p]\}$ be the ideal of all maps acting height increasing on the socle of A, and let $C(A)$ be the ideal of all elements of $\text{End}(A)$ mapping each Cauchy sequence in $A[p]$ onto a convergent one. (For $x \in A$, $|x|$ denotes the p-height of x in A and topological notations refer to the p-adic topology.) If A is torsion-complete, $J(\text{End}(A)) = H(A)$, if A is Σ-cyclic or ω_1-separable, $J(\text{End}(A)) = H(A) \cap C(A)$, and $H(A) \cap C(A) \subset J(\text{End}(A))$ for all separable p-groups (cf. [**S**]). The purpose of this paper is to show that $J(\text{End}(A))$ is in general *not* equal to $C(A) \cap H(A)$ for separable p-groups A. We will use that $J(\text{End}(A)) \cap E_s(A) = E_s(A) \cap H(A)$, where $E_s(A)$ is the ideal of all small endomorphisms of A (cf. [**S**]). Recently, many complicated p-groups have been constructed in [**DG1**, **DG2**, **CG**]. All these groups enjoy the property that $\text{End}(A)$ is a split extension of $E_s(A)$, i.e. $\text{End}(A) = R \oplus E_s(A)$ for some subring R of $\text{End}(A)$. The way these groups are constructed, $R \cap H(A) = pR$ and $\overline{H}_R(A) = \overline{H}^*_R(A)$, i.e. if $r \in R - H(A)$, then for all n there is $0 \neq x \in p^n A[p]$ such that x and xr have the same height. In this situation Theorem 1 below implies

$$J(\text{End}(A)) = (J(R) \cap H(A)) \oplus (E_s(A) \cap H(A))$$

and we have $J(\text{End}(A)) = H(A) \cap C(A)$ for these groups. We will construct a ring R and use the realization result in [**C**] to obtain a separable p-group A such that

Received by the editors July 17, 1986 and, in revised form, December 18, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 20K10, 20K30.
Key words and phrases. Abelian p-group, endomorphism ring, small endomorphisms, Jacobson radical, height-preserving maps.

$\mathrm{End}(A) = R \oplus E_s(A)$ and $(J(R) - pR) \cap H(A)$ and $(R - J(R)) \cap H(A)$ are both not empty. Since $C(A) \cap R = pR$ for this group A, we have that $H(A) \cap C(A)$ is a proper subset of $J(\mathrm{End}(A))$. This makes it hard to believe that Pierce's question has a positive answer for *all* separable p-groups.

The construction. Let A be a separable p-group and $E = \mathrm{End}(A)$ the endomorphism ring of A. If $S \subset E$, and $U \subset A[p]$ are subsets, define $\overline{H}_S(A, U) = \{\varphi \in S \mid \exists 0 \neq x \in U, |x| = |x\varphi|\}$. Observe that $\overline{H}_E(A, U) \cap S = \overline{H}_S(A, U)$ and if $U \subset V$ we have $\overline{H}_S(A, U) \subset \overline{H}_S(A, V)$. We also define $H(A) = E - \overline{H}_E(A, A[p])$, $\overline{H}_S(A) = \overline{H}_S(A, A[p])$ and $\overline{H}^*_S(A) = \bigcap_n \overline{H}_S(A, p^n A[p])$. Moreover, let $H_S(A) = S \cap H(A)$ and $H_S(A, U) = S - \overline{H}_S(A, U)$.

THEOREM 1. *Let A be a separable p-group such that $E = E(A) = R \oplus E_s(A)$ is a split extension of $E_s(A)$ and $\overline{H}_R(A) = \overline{H}^*_R(A)$. Then $J(E) = H_{J(R)}(A) \oplus H_{E_s(A)}(A)$.*

PROOF. Let $r \in R$, and $\sigma \in E_s(A)$ with $r + \sigma \in J(E)$. Then $(r + \sigma)t$ is right quasi-regular for all $t \in R$ and, since $E = R \oplus E_s(A)$, the element rt is right quasi-regular in R as well as $r \in J(R)$. Now suppose $r \notin H_{J(R)}(A)$. Since σ is small and $\overline{H}_R(A) = \overline{H}^*_R(A)$, we find $n < \omega$ and $0 \neq x \in p^n A[p]$ such that $x\sigma = 0$ and $|x| = |xr|$. Let F be a finite summand of A containing xr and let $\rho\colon A \to F$ be the natural projection. Then $xr = xr\rho$ and $|x| = |xr| = |xr\rho| = |x(r + \sigma)\rho|$. Since $A\rho = F$ is finite, $\rho \in E_s(A)$ and hence $(r+\sigma)\rho \in J(E) \cap E_s(A) = E_s(A) \cap H(A)$ (cf. [**S**]). This contradicts the above equation of heights and we conclude $r \in H_{J(R)}(A)$. Now let $t \in R$, $\sigma \in E_s(A)$ and $r \in H_{J(R)}(A)$. Then there is $s \in R$ such that $(1 - rt)s = 1$. This implies $(1 - r(t + \sigma))s = (1 - rt)s - r\sigma s = 1 - r\sigma s$. Since $\sigma \in E_s(A)$ and $r \in H_{J(R)}(A)$ we have that $r\sigma s \in H_{E_s(A)}(A) \subset J(E)$ and there is $\tau \in E$ with $(1 - r\sigma s)\tau = 1$. This implies $(1 - r(t+\sigma))s\tau = 1$ and $r \in J(E)$. We obtain $H_{J(R)}(A) \subset J(E) \subset H_{J(R)} \oplus E_s(A)$ which together with $J(E) \cap E_s(A) = H_{E_s(A)}(A)$ implies the desired equation.

We now construct our ring:

Let ω be the set of natural numbers including 0 and let

$$B = \bigoplus_{i \in \omega} \langle f_i \rangle \oplus \bigoplus_{i \in \omega} \langle g_i \rangle \oplus \bigoplus_{i \in \omega} \langle h_i \rangle$$

be a Σ-cyclic p-group with $\exp(f_i) = i + 1 = \exp(g_i)$ and $\exp(h_i) = i + 2$. We define elements $\alpha, \beta, \gamma \in \mathrm{End}(B)$ by setting $f_i\alpha = pf_{i+1}$, $f_i\beta = g_i$ and $f_i\gamma = ph_i$, and α, β and γ are 0 on the g_i's and h_i's. Let $S = \langle 1, \alpha, \beta, \gamma \rangle$ be the subring of $\mathrm{End}(B)$ generated by these elements and $R = \widehat{S}$ be the p-adic completion of S. We have the following relations:

(1) $\beta\alpha = \gamma\alpha = \beta^2 = \gamma^2 = \beta\gamma = \gamma\beta = 0$.

Each element $r \in S$ has a unique representation:

(2) $r = \sum_{i=0}^n \alpha^i a_i + \sum_{i=0}^m \alpha^i \beta b_i + \sum_{i=0}^k \alpha^i \gamma c_i$ with a_i, b_i and c_i integers.

Therefore each element $x \in R = \widehat{S}$ has a unique representation:

(3) $x = \sum_{i=0}^\infty \alpha^i a_i + \sum_{i=0}^\infty \alpha^i \beta b_i + \sum_{i=0}^\infty \alpha^i \gamma c_i$ where $\{a_i\}$, $\{b_i\}$ and $\{c_i\}$ are p-adic zero-sequences in J_p, the ring of p-adic integers. Let I be the set of all $x \in R$ with all a_i's being 0. This is the ideal of R generated by β and γ. An easy computation shows:

(4) Let $x \in R$ be as in (3). Then $\exp(f_k x) = \max_{i\in\omega}\{k+1-|a_i|,\ k+1-|b_i|,\ k+1-|c_i|\}$. Here the max is defined to be 0 if all numbers in the set are < 0. Because of (3), $\{\alpha^i \mid i < \omega\}$ is linearly independent and we have

(5) $R/I \cong (J_p[\alpha])\hat{}$, the p-adic completion of the polynomial ring $J_p[\alpha]$.

This implies

(6) $R/(pR+I) \cong GF(p)[\alpha]$, the polynomial ring over $GF(p)$.

This implies $pR \subset J(R) \subset pR + I$.

(7) Let $j < k+1$. Then $|p^j f_k x| > j = |p^j f_k|$ for all $x \in I$, the ideal generated by β and γ, i.e. $I \subset H(\overline{B})$.

This follows from a straightforward computation using that B is Σ-cyclic. Observe that $I^2 = 0$.

(8) $J(R) = pR + I$.

We want R to be pure in $E(\overline{B})$, $\overline{B}$ the torsion-completion of B. We show a little more:

(9) $R \oplus E_s(\overline{B})$ is pure in $E(\overline{B})$. (One needs this to do a "Black Box" construction; cf. [**CG**].)

To prove (9), let $\varphi \in E(\overline{B})$, $r \in R$, $\sigma \in E_s(\overline{B})$ and $n \in \omega$ with $p^n\varphi = r+\sigma$. Since σ is small, there is $k \in \omega$ such that $p^{k-n-1} f_k p^n \varphi = p^{k-n-1} f_k r$ and $p^2 p^{k-1} f_k \varphi = 0$. Hence $p^{k-n+1} f_k r = 0$. Now let r be represented as in (3) and apply (4) to obtain:

$$k-n+1 \geq \exp(f_k r) = \max_{i\in\omega}\{k+1-|a_i|, k+1-|b_i|, k+1-|c_i|\}.$$

This implies $k-n+1 \geq k+1-|a_i|$ and $|a_i| \geq n$. The same holds for the b_i's and c_i's. Therefore $r \in p^n R$ and $r = p^n s$ for some $s \in R$. Thus $p^n(\varphi - s) = \sigma$ is small which implies $\varphi - s$ is small and $\varphi \in R \oplus E_s(A)$. □

(10) Let A be a pure subgroup of $\overline{B}$ containing B and $\varphi \in H_{E(A)}(A, B[p])$. Then $\varphi \in H_{E(A)}(A, A[p])$.

Let $a \in A[p] - B[p]$. Since A/B is divisible, there is $b \in B$, $y \in A$ such that $a = b + p^{n+1}y$ where $n = |a|$. Then $|a| = |b|$ and $|a\varphi| = |(b+p^{n+1}y)\varphi| \geq n+1 > n = |a|$ since $\varphi \in H_{E(A)}(A, B[p])$. This inequality shows $\varphi \in H_{E(A)}(A, A[p])$. □

Now we apply A. L. S. Corner's result [**C**, Theorem 2.1] and obtain a pure subgroup A of $\overline{B}$ containing B and $\mathrm{End}(A) = R \oplus E_s(A)$. (Observe that (4) implies that condition (C) of [**C**, Theorem 2.1] holds.) The ring R is constructed to satisfy $H_R(A) = pR + \alpha R + R\gamma$. Moreover we have $\alpha \in h_R(A) - J(R)$, $\beta \in J(R) - H_R(A)$ and $\gamma \in H_{J(R)}(A) - pR$. Observe that $\gamma \notin C(A)$, since otherwise $\overline{B}[p]\gamma \subset A$. In order to see that this is absurd, we have to look into Corner's proof [**C**] of his Theorem 2.1: Recall that for a positive integer e an element $x \in \overline{B}$ is e-strong if $xr = 0$ implies $r \in p^e R$ for $x \in \overline{B}[p^e]$ and $r \in R$. Our group A is one of Corner's G_ρ (cf. [**C**, Theorem 2.1]). Corner observes [**C**, p. 285, line -6] that each of his G_ρ contains for any e an e-strong element x such that for $\sigma \neq \rho$ we have $G_\sigma \cap xR = 0$. For $e = 1$, $x \in \overline{B}[p]$ and since $\gamma \notin pR$ we conclude $x\gamma \neq 0$. This shows that $\overline{B}[p]\gamma$ is not contained in any G_σ $(= A)$. Now Theorem 1 applies and we have that $J(\mathrm{End}(A)) = H_{J(R)}(A) \oplus H_{E_s(A)}(A)$ is *not* contained in $C(A) \cap H(A)$ since γ is not and also $H_R(A)$ is not contained in $J(R)$. So if we want to describe the elements of $J(\mathrm{End}(A))$ by their action on A, we have to find the elements in $J(R) \cap H(A)$, which means we must be able to recognize the elements of $J(R)$. There is much freedom for the way an element of $J(R)$ can operate on A. We answer a question in [**S**] by summarizing part of our discussion in

THEOREM 2. *There exists a separable p-group A such that* $J(\mathrm{End}(A))$ *is larger than* $H(A) \cap C(A)$.

REMARK. If we want to have larger groups A realizing R, we may employ Shelah's "Black Box" and a construction very similar to the one in [**CG**]. We would like to mention again that all the p-groups constructed in [**CG, DG1** or **DG2**] satisfy $H(A) \cap J(R) = pR$.

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