PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 102, Number 4, April 1988

L_p-CONTINUITY OF POSITIVE SEMIGROUPS ON FINITE VON NEUMANN ALGEBRAS

SEIJI WATANABE

(Communicated by John B. Conway)

ABSTRACT. Let M be a σ-finite, finite von Neumann algebra with a faithful normal tracial state τ. Let α be a one-parameter semigroup of normal positive contractions of M. Then it is shown that α is continuous with respect to the L_p-norm $(1 \leq p < \infty)$ induced by τ if and only if it is σ-weakly continuous.

1. Introduction. Various useful topologies are introduced on von Neumann algebras, including the (operator) norm topology. Hence, various continuity properties can also be defined for one-parameter semigroups of linear operators. However, only σ-weakly continuous semigroups or groups have been discussed in most papers, except uniformly continuous ones. Indeed, R. R. Kallman [**5, 6**] showed that every strongly continuous (namely, continuous in the pointwise norm convergence topology) one-parameter group of *-automorphisms of von Neumann algebras is already uniformly continuous, and G. A. Elliott [**2**] showed the same result for AW*-algebras. Furthermore, the uniform continuity of strongly continuous positive semigroups was shown by A. Kishimoto and D. W. Robinson [**7**] for abelian von Neumann algebras and by U. Groh [**4**] for properly infinite von Neumann algebras. Nevertheless, the problem for arbitrary von Neumann algebras remains open.

In this paper, we examine L_p-continuity of positive semigroups on finite von Neumann algebras, using a technique of [**3, 8**].

2. Result. Let M be a σ-finite, finite von Neumann algebra with a fixed faithful normal tracial state τ. Let $\| \ \|_p$ $(1 \leq p < \infty)$ be the L_p-norm on M defined by $\|x\|_p = \tau(|x|^p)^{1/p}$ $(x \in M)$ and $\| \ \|_\infty$ be the (operator) norm on M. Let $\alpha = \{\alpha_t\}_{t \geq 0}$ be a one-parameter semigroup of normal positive contractions on M with $\alpha_0 = I$ (the identity operator). Then α is said to be L_p-norm continuous if $\lim_{t \downarrow 0} \|\alpha_t(x) - x\|_p = 0$ for every $x \in M$ $(1 \leq p \leq \infty)$.

We shall consider L_p-continuity of α in the following. If α is L_p-norm continuous for some p $(1 \leq p \leq \infty)$, then α is σ-weakly continuous. Indeed, since τ is faithful and every α_t is contraction with respect to $\| \ \|_\infty$, $|\tau(\alpha_t(x) - x)| \leq \|\alpha_t(x) - x\|_p$ $(x \in M)$ implies that $\rho(\alpha_t(x)) \to \rho(x)$ $(t \to +0)$ for every $x \in M$ and $\rho \in M_*$ (the predual space of M). The converse statement does not hold for $p = \infty$ in general as mentioned above. But it does hold for p $(1 \leq p < \infty)$. Namely, we have the following theorem. This is interesting in connection with the problem mentioned in the Introduction and the fact that $\lim_{p \to \infty} \|x\|_p = \|x\|_\infty$ for $x \in M$.

Received by the editors December 19, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46L10, 46L50, 47D05.
Key words and phrases. Positive semigroup, trace, finite von Neumann algebra.

THEOREM. *Let M, τ, and α be as above. Then α is L_p-norm continuous for every p $(1 \leq p < \infty)$ if and only if α is σ-weakly continuous.*

Before going to the proof, we should note that τ is not necessarily assumed to be α-invariant.

PROOF OF THEOREM. The if part is already seen. Conversely, suppose that α is σ-weakly continuous. Let $\beta = \{\beta_t\}_{t\geq 0}$ be the strongly continuous contraction semigroup on $L^1(M,\tau)$ corresponding to the preadjoint semigroup of α on M_*, where $L^1(M,\tau)$ is the noncommutative L^1-space realized as the space of closed operators [**1**, **9**]. We define two positive elements $\tau_0 \in M_*$ and $h \in L^1(M,\tau)$ by

$$\tau_0 = \int_0^\infty e^{-t}\tau \circ \alpha_t \, dt$$

and

$$h = \int_0^\infty e^{-t}\beta_t(1)\, dt.$$

Then τ_0 is faithful and h is injective because $\alpha_0 = I$ and $\tau_0(\cdot) = \tau(h\cdot)$. Now, an inner product $\langle\ ,\ \rangle$ on M is defined by $\langle x, y\rangle = \tau_0(y^*x + xy^*)$ $(x, y \in M)$, and let H be the completion of M in the norm $|\!|\!| x |\!|\!|_2 = \langle x, x\rangle^{1/2}$ $(x \in M)$. Next, we put $\gamma_t(x) = e^{-t}\alpha_t(x)$ $(x \in M,\ t \geq 0)$. Then $\gamma = \{\gamma_t\}_{t\geq 0}$ is a σ-weakly continuous semigrup of normal positive contractions of M, and we have, for $x \in M$,

$$\begin{aligned}
\tau_0(\gamma_t(x^*x + xx^*)) &= \int_0^\infty e^{-s}\tau(\alpha_s(e^{-t}\alpha_t(x^*x + xx^*)))\, ds \\
&= \int_t^\infty e^{-s}\tau(\alpha_s(x^*x + xx^*))\, ds \\
&\leq \int_0^\infty e^{-s}\tau(\alpha_s(x^*x + xx^*))\, ds \\
&= \tau_0(x^*x + xx^*).
\end{aligned}$$

Since each γ_t is a positive $\|\ \|_\infty$-contraction on M, we have, for $x \in M$,

$$\begin{aligned}
|\!|\!| \gamma_t |\!|\!|_2^2 &= \tau_0(\gamma_t(x)^*\gamma_t(x) + \gamma_t(x)\gamma_t(x)^*) \\
&\leq \tau_0(\gamma_t(x^*x + xx^*)) \\
&\leq \tau_0(x^*x + xx^*) = |\!|\!| x |\!|\!|_2^2.
\end{aligned}$$

That is, each γ_t is contraction on M with respect to the norm $|\!|\!|\ |\!|\!|_2$. Hence, each γ_t extends to a contraction, say $\tilde{\gamma}_t$, on H. Thus, we have a contraction semigroup on H which is weakly continuous because γ is σ-weakly continuous on M. Hence, $\tilde{\gamma} = \{\tilde{\gamma}_t\}_{t\geq 0}$ is strongly continuous on H, that is,

$$\lim_{t\downarrow 0} |\!|\!| \tilde{\gamma}_t(x) - x |\!|\!|_2 = 0 \qquad (x \in H).$$

By a simple calculation, we have $\lim_{t\downarrow 0} |\!|\!| \alpha_t(x) - x |\!|\!|_2 = 0$ $(x \in M)$. Next, let $h = \int_0^\infty \lambda\, de(\lambda)$ be the spectral resolution of h. Then $\tau(e(\lambda)) \to \tau(e(0)) = 0$ by the injectivity of h. Therefore, for each positive number $\varepsilon > 0$, there exists a positive number $\lambda > 0$ such that $\tau(e(\lambda)) < \varepsilon$. Then we have $\lambda(1 - e(\lambda)) \leq h$. Thus, we have, for $x \in M$,

$$\begin{aligned}\|\alpha_t(x) - x\|_2^2 &= \tau(|\alpha_t(x) - x|^2) \\ &= \tau((1 - e(\lambda))|\alpha_t(x) - x|^2) + \tau(e(\lambda)|\alpha_t(x) - x|^2) \\ &= \tau(|\alpha_t(x) - x|(1 - e(\lambda))|\alpha_t(x) - x|) \\ &\quad + \tau(e(\lambda)|\alpha_t(x) - x|^2 e(\lambda)) \\ &\leq (1/\lambda)\tau(|\alpha_t(x) - x|h|\alpha_t(x) - x|) + (2\|x\|_\infty)^2\tau(e(\lambda)) \\ &\leq (1/\lambda)\tau(h|\alpha_t(x) - x|^2) + (2\|x\|_\infty)^2\varepsilon \\ &= (1/\lambda)\tau_0(|\alpha_t(x) - x|^2) + 4\|x\|_\infty^2\varepsilon \\ &\leq (1/\lambda)|\!|\!|\alpha_t(x) - x|\!|\!|_2^2 + 4\|x\|_\infty^2\varepsilon.\end{aligned}$$

Since $|\!|\!|\alpha_t(x) - x|\!|\!|_2^2 \to 0$ $(t \downarrow 0)$ and ε is arbitrary, we have $\lim_{t\downarrow 0} \|\alpha_t(x) - x\|_2 = 0$ $(x \in M)$. Furthermore, inequalities $\|a\|_p \leq \|a\|_\infty^{(p-1)/p}\|a\|_2^{1/p}$ $(a \in M,\ \infty > p \geq 1)$ imply our assertion. The proof is completed.

REFERENCES

1. J. Dixmier, *Formes lineaires sur un anneau d'operateurs*, Bull. Soc. Math. France **81** (1953), 9–39.
2. G. A. Elliott, *Convergence of automorphisms in certain C^*-algebras*, J. Funct. Anal. **11** (1972), 204–206.
3. R. Emilion, *Semi-groups in L_∞ and local ergodic theorem*, Canad. Math. Bull. **29** (1986), 146–153.
4. U. Groh, *Positive semi-groups on C^*- and W^*-algebras*, Lecture Notes in Math., vol. 1184, Springer-Verlag, Berlin and New York, 1986, pp. 369–425.
5. R. R. Kallman, *Unitary groups and automorphisms of operator algebras*, Amer. J. Math. **91** (1969), 785–806.
6. ——, *One-parameter groups of *-automorphisms of* II_1 *von Neumann algebras*, Proc. Amer. Math. Soc., **24** (1970), 336–340.
7. A. Kishimoto and D. W. Robinson, *Subordinate semi-groups and order properties*, J. Austral. Math. Soc. Ser. A **31** (1981), 59–76.
8. ——, *On unbounded derivations commuting with a compact group of *-automorphisms*, Publ. Res. Inst. Math. Sci. **18** (1982), 1121–1136.
9. I. Segal, *A noncommutative extension of abstract integration*, Ann. of Math. (2) **57** (1953), 401–457.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, NIIGATA UNIVERSITY, NIIGATA 950-21, JAPAN