A DECOMPOSITION OF BOUNDED SCALARLY MEASURABLE FUNCTIONS TAKING THEIR RANGES IN DUAL BANACH SPACES

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ABSTRACT. A decomposition of scalarly measurable functions taking their range in the dual of a Banach space into a Pettis integrable part and a weak* scalarly null part is introduced and analyzed.

1. Introduction. It is shown in [11] that if $X$ is a separable Banach space, $K$ is a compact Hausdorff space, and $f : K \to X^*$ is a bounded universally scalarly measurable function, then $f$ is universally Pettis integrable. In order to prove this, they first show the following:

**Lemma.** Let $(\Omega, \Sigma, \mu)$ be a finite measure space and let $X$ be a separable Banach space. If $f : \Omega \to X^*$ is a bounded scalarly measurable function such that for every $\varepsilon > 0$, there exists $E \in \Sigma$ such that $\mu(\Omega \setminus E) < \varepsilon$ and $\{(f, x) : x \in E, \|x\| < 1\}$ is weakly precompact in $L^\infty(\mu)$, then $f$ is $\mu$-Pettis integrable.

They also point out that the separability hypothesis in both of these cannot be removed. In particular there is an example due to Phillips (see Example B in the next section) of a function $f : [0,1] \to l^\infty[0,1]$ such that $f$ is scalarly Borel measurable but not Lebesgue Pettis integrable.

In [8] Musial shows that if $X$ is a separable Banach space then the following statements are equivalent:

(i) $X$ does not contain any isomorphic copy of $l^1$.
(ii) $X^*$ has the weak Radon Nikodym property.
(iii) If $(\Omega, \Sigma, \mu)$ is a complete measure space and $f : \Omega \to X^*$ is bounded and weak* scalarly measurable, then $f$ is Pettis integrable.
(iv) If $(\Omega, \Sigma, \mu)$ is a complete measure space and $f : \Omega \to X^*$ is bounded and weak* scalarly measurable, then $f$ is scalarly measurable.

Janika was able to extend the equivalence of statements (i) and (ii) for the case $X$ not necessarily separable in [7]. However, when we remove the separability hypothesis, statements (i) and (ii) are not equivalent to statements (iii) and (iv) as seen in Example C in the next section.

The purpose of this paper is to extend the above-mentioned results to the case $X$ not necessarily separable by a suitable weakening of the conclusion.

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Let us introduce some definitions and preliminary facts. Let $(\Omega, \Sigma, \mu)$ be a finite measure space and $X$ be a Banach space. A function $f: \Omega \to X^*$ is scalarly measurable if $x^{**} \circ f$ is measurable for every $x^{**} \in X^{**}$, and $f$ is weak* scalarly measurable if $x \circ f$ is measurable for every $x \in X$. The function $f$ is $\mu$-Pettis integrable if for every $E \in \Sigma$, there exists $x^*_E \in X^*$ such that

$$x^{**}(x^*_E) = \int_E x^{**} \circ f \, d\mu$$

for every $x^{**} \in X^{**}$.

In this case, $x^*_E = \mu \int_E f \, d\mu$.

If $K$ is a compact Hausdorff space, then $f: K \to X^*$ is universally scalarly measurable if $f$ is scalarly measurable with respect to all Radon measures on $K$.

A weak* compact subset $K$ of $X^*$ is called a Pettis set if the canonical injection $K \to X^*$ is scalarly measurable for each Radon measure on $(K, \text{weak}^*)$.

A subset $K$ of a Banach space $X$ is called weakly precompact if every bounded sequence has a weakly Cauchy subsequence. Rosenthal's theorem [10] states that a bounded subset $K$ of a Banach space $X$ is weakly precompact if and only if $K$ does not contain an isomorphic copy of the basis of $l^1$.

A Banach space $X$ has the weak Radon Nikodym property (WRNP) if for every finite measure space $(\Omega, \Sigma, \mu)$, every operator $T: L^1(\mu) \to X^*$ is Pettis representable, i.e. there exists a Pettis integrable function $f$ such that $T(g) = \mu \int f \, g \, d\mu$.

2. Weak* scalarly null functions. Let $X$ be a Banach space and let $(\Omega, \Sigma, \mu)$ be a finite measure space. We define a function $f: \Omega \to X^*$ to be $\mu$-weak* scalarly null if $x \circ f = 0$ a.e.-[\mu] for every $x \in X$.

**Proposition 1.** Let $X$ be a separable Banach space. Then $f$ is $\mu$-weak* scalarly null if and only if $f = 0$ a.e.-[\mu].

**Proof.** Let $(x_j)_{j=1}^\infty$ be dense in $X$. Let $N_j$ be $\mu$-null sets such that $x_j \circ f(t) = 0$ if $t \in \Omega \setminus N_j$. Clearly $f(t) = 0$ if $t \in \Omega \setminus (\bigcup_{j=1}^\infty N_j)$.

Consequently any interesting example of a weak* scalarly null function will occur only if the function takes its range in a nonseparable space.

**Examples.** In each of the following examples, $\mu$ is Lebesgue measure on $[0,1]$ and $\Sigma$ is the Lebesgue measurable subsets of $[0,1]$.

**Example A.** Let $X = l^2[0,1]$. Define $f: [0,1] \to l^2[0,1] = X^*$ by

$$f(s)(t) = \begin{cases} 1 & \text{if } s = t, \\ 0 & \text{if } s \neq t. \end{cases}$$

If $x \in l^2[0,1]$ then $x \circ f$ is countably nonzero since $x$ has countable support. Therefore, $f$ is $\mu$-weak* scalarly null. It is also easy to see that $f$ is $\mu$-Pettis integrable and that $\mu \int_E f \, d\mu = 0$ for every $E \in \Sigma$, even though $f$ is not equal to zero a.e.-[\mu].

**Example B (Phillips).** Let $X = l^1[0,1]$. Let $B$ be the subset of $[0,1] \times [0,1]$ constructed by Sierpinski having the following properties:

(i) for every $t_0 \in [0,1]$, $\{s: (s,t_0) \in B\}$ is countable.

(ii) for every $s_0 \in [0,1]$, $\{t: (s_0,t) \notin B\}$ is countable.

Define $f: [0,1] \to l^\infty[0,1]$ by $f(s)(t) = \chi_B(s,t)$. It is shown in [5] that $f$ is bounded and scalarly Borel measurable but not $\mu$-Pettis integrable. If $x \in l^1[0,1]$, it is clear that $x \circ f$ is countably nonzero and $f$ is $\mu$-weak* scalarly null.
Example C. Let \( f : [0,1] \to l^1 [0,1] \) be given by \( f(t) = e_t \). If \( x \in c_0 [0,1] \), then again \( x \circ f \) is countably nonzero and \( f \) is \( \mu \)-weak* scalarly null. However, if \( A \) is any subset of \([0,1]\), then \( \chi_A \in l^\infty [0,1] \) and \( \chi_A \circ f = \chi_A \). Hence \( f \) is not scalarly measurable.

The above examples show that a nontrivial weak* scalarly null function may be Pettis integrable, scalarly measurable but not Pettis integrable, or even not scalarly measurable.

**Proposition 2.** Let \( X \) be a Banach space and \( (\Omega, \Sigma, \mu) \) be a finite measure space. If \( f : \Omega \to X^* \) is bounded and scalarly measurable then the following are equivalent:

(i) There exists a \( \mu \)-Pettis integrable function \( g \) and a \( \mu \)-weak* scalarly null function \( h \) such that \( f = g + h \).

(ii) There exists a \( \mu \)-Pettis integrable function \( g \) such that for every \( x^{**} \in X^{**} \), \( T_f^{**}(x^{**}) = x^{**} \circ g \) in \( L^1(\mu) \).

(iii) For every \( \varepsilon > 0 \), there exists \( A \in \Sigma \) and a Pettis integrable function \( g \) such that \( \mu(\Omega \setminus A) < \varepsilon \) and \( (x \circ f)\chi_A = x \circ g \) a.e.-[\( \mu \)] for every \( x \in X \).

**Proof.** (i)\( \to \) (ii) If \( f = g + h \) and \( h \) is weak* scalarly null, then \( x \circ f = x \circ g \) in \( L^1(\mu) \) for every \( x \in X \). Since \( g \) is Pettis integrable it is clear that \( T_g^{**}(x^{**}) = x^{**} \circ g \) in \( L^1(\mu) \) for every \( x^{**} \in X^{**} \). Hence we have that for every \( x^{**} \in X^{**} \),

\[
T_f^{**}(x^{**}) = T_g^{**}(x^{**}) = x^{**} \circ g \quad \text{in } L^1(\mu).
\]

(ii)\( \to \) (iii) Clear.

(iii)\( \to \) (i) Use a standard exhaustion argument as in Lemma III.2.4 in [3] to get a Pettis integrable function \( g \) such that \( x \circ f = x \circ g \) a.e.-[\( \mu \)] for every \( x \in X \). Clearly \( h = f - g \) is \( \mu \)-weak* scalarly null.

We say a bounded scalarly measurable function is \( \mu \)-Pettis decomposable if any and hence all of the above statements are true. An immediate consequence of Proposition 1 is that if \( X \) is a separable Banach space then a bounded scalarly measurable function \( f : \Omega \to X^* \) is Pettis decomposable if and only if \( f \) is Pettis integrable.

**3. Decomposition and the RS-property.** In [13], Talagrand defines a weak* measurable function \( f : \Omega \to X^* \) to have the RS-property if the Radon image measure \( \nu = \mu \circ f^{-1} \) is such that for every \( n \) there is a Pettis set \( K_n \) such that \( \nu(\Omega \setminus K_n) \leq 1/2^n \). He then shows that if \( f \) is weak* scalarly bounded, then \( f \) has the RS-property if and only if for every \( \varepsilon > 0 \) there exists \( E \in \Sigma \) with \( \mu(\Omega \setminus E) < \varepsilon \) such that the set \( \{(f, x) \in \Sigma : \|x\| \leq 1\} \) is weakly precompact. Therefore the Lemma of Riddle, Saab, and Uhl mentioned in the introduction can be stated as follows: If \( X \) is a separable Banach space and \( f : \Omega \to X^* \) is bounded, scalarly measurable and has the RS-property, then \( f \) is Pettis integrable. We are now able to extend this result.

**Theorem 3.** Let \( (\Omega, \Sigma, \mu) \) be a finite measure space and \( X \) a Banach space. If \( f : \Omega \to X^* \) is a bounded scalarly measurable function such that \( f \) has the RS-property, the \( f \) is \( \mu \)-Pettis decomposable.

**Proof.** By Theorem 7-3-15 of [13], if \( f \) has the RS-property, then there exists a Pettis integrable function \( g : \Omega \to X^* \) such that \( x \circ f = x \circ g \) in \( L^1(\mu) \) for every
In Proposition 7-3-16 of [13], Talagrand exhibits a Pettis integrable function \( \varphi : [0,1] \to \ell^\infty \) that does not have the RS-property. Obviously \( \varphi \) is Pettis decomposable. Hence it is clear that decomposability is strictly weaker than the RS-property.

Talagrand raises the following question: If \( K \) is a compact Hausdorff space and \( f : K \to X^* \) is universally scalarly measurable, must \( f \) have the RS-property for every Radon measure \( \mu \) on \( K \)? It is also natural to ask the following:

**QUESTION.** If \( K \) is a compact Hausdorff space, \( X \) is a Banach space, and \( f : K \to X^* \) is bounded and universally scalarly measurable, must \( f \) be \( \mu \)-decomposable for every Radon measure \( \mu \)?

In [10], Riddle and Saab show that the answer to Talagrand’s question is yes if \( f \) is universally Lusin measurable, and in fact such an \( f \) must be Pettis integrable.

4. Pettis decomposition and the weak Radon Nikodym property. We begin this section with the following observation.

**Lemma 4.** Let \( (\Omega, \Sigma, \mu) \) be a perfect measure space and \( f : \Omega \to X^* \) be a bounded weak-star scalarly measurable function. If \( f = g + h \), where \( g \) is scalarly measurable and \( h \) is weak-star scalarly null, then the operator \( T_f : X \to L^1(\mu) \), defined by \( T_f(x) = x \circ f \) for every \( x \in X \), is compact.

**Proof.** Since \( h \) is weak-star scalarly null, \( T_f(x) = T_g(x) \) in \( L^1(\mu) \) for every \( x \in X \). However, since \( g \) is scalarly measurable, the operator \( T_g \) is compact by Proposition 3 of [1].

We are now able to prove the following extension of Musial’s result.

**Theorem 5.** If \( X \) is a Banach space, then the following are equivalent:

(i) \( X \) does not contain an isomorphic copy of \( \ell^1 \).

(ii) \( X^* \) has the WRNP.

(iii) If \( (\Omega, \Sigma, \mu) \) is a complete measure space and \( f : \Omega \to X^* \) is bounded and weak-star scalarly measurable, then \( f \) is Pettis decomposable.

(iv) If \( (\Omega, \Sigma, \mu) \) is a complete measure space and \( f : \Omega \to X^* \) is bounded and weak-star scalarly measurable, then \( f = g + h \), where \( g \) is scalarly measurable and \( h \) is weak-star scalarly null.

**Proof.** Janika proved the equivalence of (i) and (ii) in [7]. The fact that (ii) implies (iii) is given in Theorem 2 of [8]. We have only left to show (iv) implies (i).

It is shown in [2, Theorem 3.5] that if \( X \) is a Banach space that contains \( \ell^1 \), then there exists a bounded weak-star Lebesgue measurable function \( f : [0,1] \to X^* \) such that the operator \( T_f : X \to L^1([0,1]) \) is not compact. Hence by Lemma 4, this function cannot satisfy the conclusion of statement (iv).

**Addendum.** The question in §3 has been affirmatively answered by the author in “Pettis decomposition for universally scalarly measurable functions”.

**References**


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