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ON THE MODULUS OF WEAKLY COMPACT OPERATORS AND STRONGLY ADDITIVE VECTOR MEASURES

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ABSTRACT. Aliprantis and Burkinshaw proved that each weakly compact operator from an AL-space into a KB-space has a weakly compact modulus. In the present paper it is shown that this is also true for weakly compact operators from a Banach lattice having an order continuous dual norm into an order complete AM-space with unit. A corresponding result is obtained for strongly additive vector measures.

1. Introduction. Throughout this paper, let $\mathbf{E}$ and $\mathbf{G}$ be Banach lattices and let $\mathcal{F}$ be an algebra of subsets of some set Ω. A (linear) operator $T\colon \mathbf{E} \to \mathbf{G}$ is *weakly compact* if it maps the closed unit ball of $\mathbf{E}$ into a relatively weakly compact subset of $\mathbf{G}$, and a vector measure $\mu\colon \mathcal{F} \to \mathbf{G}$ is *strongly additive* if the series $\sum \mu(A_n)$ converges for each sequence $\{A_n | n \in \mathbf{N}\}$ of mutually disjoint sets in $\mathcal{F}$. It is known that a vector measure is strongly additive if and only if its range is relatively weakly compact [**3**, Theorem I.5.2].

In the present paper we give conditions under which the ordered vector spaces of all weakly compact operators $\mathbf{E} \to \mathbf{G}$ and of all strongly additive vector measures $\mathcal{F} \to \mathbf{G}$ are vector lattices. Our starting point is the following result of Aliprantis and Burkinshaw [**1**] (see also [**2**, Theorem 17.14]):

1.1. PROPOSITION. *If $\mathbf{E}$ is an AL-space and $\mathbf{G}$ is a KB-space, then each weakly compact operator $T\colon \mathbf{E} \to \mathbf{G}$ has a weakly compact modulus $|T|$.*

A comparison of Proposition 1.1 with corresponding results for bounded operators [**4**, Theorem IV.1.5] and compact operators [**4**, Theorem IV.4.6 Corollary 2] suggests that a result analogous to Proposition 1.1 should hold for weakly compact operators taking their values in an order complete AM-space with unit. In §2 of this paper, we shall prove the following result:

1.2. THEOREM. *If $\mathbf{E}$ has an order continuous dual norm and $\mathbf{G}$ is an order complete AM-space with unit, then each weakly compact operator $T\colon \mathbf{E} \to \mathbf{G}$ has a weakly compact modulus $|T|$.*

Another motivation for Theorem 1.2 comes from measure theory: Let $\mathbf{E}_{\mathcal{F}}$ denote the sup-norm completion of the vector lattice of all $\mathcal{F}$-measurable simple functions $\Omega \to \mathbf{R}$. Then $\mathbf{E}_{\mathcal{F}}$ is an AM-space with unit, and there exists a bijection between

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the strongly additive vector measures $\mathcal{F} \to \mathbf{G}$ and the weakly compact operators $\mathbf{E}_{\mathcal{F}} \to \mathbf{G}$ [**3**, Theorem VI.1.1]. Obviously, Proposition 1.1 cannot be applied but Theorem 1.2 may be used to prove the following result:

1.3. THEOREM. *If* $\mathbf{G}$ *is an order complete AM-space with unit, then each strongly additive vector measure* $\mu\colon \mathcal{F} \to \mathbf{G}$ *has a strongly additive modulus* $|\mu|$.

In §3 of this paper, we shall give an elementary proof of Theorem 1.3 which is independent of the aforementioned representation of strongly additive vector measures by weakly compact operators. Combined with Kakutani's representation theorem for AM-spaces with unit [**4**, Theorem II.7.4 Corollary 1] and the representation of weakly compact operators $\mathcal{C}(S) \to \mathbf{G}$ by strongly additive vector measures $\mathcal{B}(S) \to \mathbf{G}$ [**3**, Theorem VI.2.5], where S is a compact Hausdorff space and $\mathcal{B}(S)$ denotes the algebra of all Borel subsets of S, Theorem 1.3 in turn may be used to prove the assertion of Theorem 1.2 in the special case where $\mathbf{E}$ and $\mathbf{G}$ are AM-spaces with unit such that $\mathbf{G}$ is order complete.

2. Weakly compact operators. For a Banach lattice $\mathbf{H}$ and $x \in \mathbf{H}_+$, let $\pi(x)$ denote the collection of all finite families $\{x_1, x_2, \ldots, x_n\}$ in $\mathbf{H}_+$ which satisfy $x = \sum x_i$.

2.1. LEMMA. *If* $\mathbf{G}$ *is order complete, then a linear operator* $T\colon \mathbf{E} \to \mathbf{G}$ *is order bounded if and only if* $\sup_{\pi(x)} \sum |Tx_i|$ *exists for all* $x \in \mathbf{E}_+$*, and in this case*

$$|T|(x) = \sup_{\pi(x)} \sum |Tx_i|$$

holds for all $x \in \mathbf{E}_+$.

PROOF. It is obvious that T is order bounded if $\sup_{\pi(x)} \sum |Tx_i|$ exists for all $x \in \mathbf{E}_+$.

Suppose now that T is order bounded. Then $|T|$ exists. Consider $x \in \mathbf{E}_+$. Then we have

$$|T|(x) = \sup_{[0,x]}(Ty - T(x-y)),$$

and we also have

$$\sum |Tx_i| \leq \sum |T|(x_i) = |T|(x)$$

for all $\{x_1, x_2, \ldots, x_n\} \in \pi(x)$. Therefore, $\sup_{\pi(x)} \sum |Tx_i|$ exists and we have

$$|T|(x) = \sup_{[0,x]}(Ty - T(x-y)) \leq \sup_{\pi(x)} \sum |Tx_i| = |T|(x),$$

which yields $|T|(x) = \sup_{\pi(x)} \sum |Tx_i|$. $\square$

For a Banach lattice $\mathbf{H}$, let $Q_{\mathbf{H}}\colon \mathbf{H} \to \mathbf{H}''$ denote the evaluation map.

2.2. LEMMA. *If there exists a positive contractive projection* $P\colon \mathbf{G}'' \to Q_{\mathbf{G}}(\mathbf{G})$*, then a linear operator* $T\colon \mathbf{E} \to \mathbf{G}$ *is order bounded if and only if its adjoint* $T'\colon \mathbf{G}' \to \mathbf{E}'$ *is order bounded, and in this case* T *satisfies*

$$|T| = Q_{\mathbf{G}}^{-1} P |T'|' Q_{\mathbf{E}}$$

and $\| \, |T| \, \| = \| \, |T'| \, \|$.

PROOF. Suppose first that T is order bounded. Then we have

$$\sum |T'x_i| = \sum |x_i T| \leq \sum x_i |T| = \sum |T|'(x_i) = |T|'(x)$$

for all $x \in \mathbf{G}'_+$ and $\{x_1, x_2, \ldots, x_n\} \in \pi(x)$, and it now follows from Lemma 2.1 that T' is order bounded and satisfies $|T'| \le |T|'$.

Suppose now that T' is order bounded. Then, by the preceding argument, T'' is order bounded and satisfies $|T''| \le |T'|'$. Furthermore, for all $x \in \mathbf{E}_+$ and $\{x_1, x_2, \ldots, x_n\} \in \pi(x)$, we have

$$\sum |Q_{\mathbf{G}} T x_i| = \sum |T'' Q_{\mathbf{E}} x_i| \le \sum |T''| Q_{\mathbf{E}} x_i = |T''| Q_{\mathbf{E}} x,$$

hence

$$Q_{\mathbf{G}} \left(\sum |T x_i| \right) = \sum |Q_{\mathbf{G}} T x_i| \le P |T''| Q_{\mathbf{E}} x,$$

and thus

$$\sum |T x_i| \le Q_{\mathbf{G}}^{-1} P |T''| Q_{\mathbf{E}} x \le Q_{\mathbf{G}}^{-1} P |T'|' Q_{\mathbf{E}} x,$$

and it now follows from Lemma 2.1 that T is order bounded and satisfies

$$|T| \le Q_{\mathbf{G}}^{-1} P |T'|' Q_{\mathbf{E}}.$$

In particular, if T is order bounded, then we have

$$|T| \le Q_{\mathbf{G}}^{-1} P |T'|' Q_{\mathbf{E}} \le Q_{\mathbf{G}}^{-1} P |T|'' Q_{\mathbf{E}} = Q_{\mathbf{G}}^{-1} P Q_{\mathbf{G}} |T| = |T|$$

and

$$\| \, |T| \, \| \le \| \, |T'|' \| = \| \, |T'| \, \| \le \| \, |T|' \| = \| \, |T| \, \|.$$

This proves the lemma. □

We can now prove our main result:

2.3. THEOREM. *If* $\mathbf{E}$ *has an order continuous dual norm and* $\mathbf{G}$ *is an order complete AM-space with unit, then each weakly compact operator* $T \colon \mathbf{E} \to \mathbf{G}$ *has a weakly compact modulus* $|T|$.

PROOF. By Gantmacher's theorem [**2**, Theorem 17.2], the adjoint $T' \colon \mathbf{G}' \to \mathbf{E}'$ is weakly compact. Since $\mathbf{G}'$ is an AL-space [**2**, Theorem 12.22] and $\mathbf{E}'$ is a KB-space [**2**, Theorem 14.11], it follows from Proposition 1.1 that T' has a weakly compact modulus $|T'|$, and using Gantmacher's theorem again we see that $|T'|'$ is weakly compact. Furthermore, since $\mathbf{G}$ is an order complete AM-space with unit, there exists a positive contractive projection $P \colon \mathbf{G}'' \to Q_{\mathbf{G}}(\mathbf{G})$ [**4**, Theorem II.7.10 Corollary 2]. Since T' is order bounded, it now follows from Lemma 2.2 that $|T|$ exists and is weakly compact. □

If $\mathbf{G}$ is order complete, then the ordered vector space $\mathcal{L}^b(\mathbf{E}, \mathbf{G})$ of all order bounded operators $\mathbf{E} \to \mathbf{G}$ is an order complete vector lattice [**2**, Theorem 1.13]. Let $\mathcal{W}(\mathbf{E}, \mathbf{G})$ denote the ordered vector space of all weakly compact operators $\mathbf{E} \to \mathbf{G}$.

2.4. COROLLARY. *If* $\mathbf{E}$ *has an order continuous dual norm and* $\mathbf{G}$ *is an order complete AM-space with unit, then* $\mathcal{W}(\mathbf{E}, \mathbf{G})$ *is an ideal in* $\mathcal{L}^b(\mathbf{E}, \mathbf{G})$.

This is immediate from Theorem 2.3 and a result of Wickstead [**7**] (see also [**2**, Theorem 17.10]).

3. Strongly additive vector measures. For a vector measure $\mu \colon \mathcal{F} \to \mathbf{G}$, let $|||\mu|||$ denote its semivariation $\mathcal{F} \to \overline{\mathbf{R}}_+$.

3.1. LEMMA. *If* $\mathbf{G}$ *is an order complete AM-space with unit, then a vector measure* $\mu\colon \mathcal{F} \to \mathbf{G}$ *is order bounded if and only if it is bounded, and in this case*

$$|||\mu|||(A) = ||| \, |\mu| \, |||(A) = || \, |\mu|(A)||$$

holds for all $A \in \mathcal{F}$.

PROOF. It is obvious that μ is order bounded if and only if it is bounded.

Suppose now that μ is order bounded and hence bounded. Then $|\mu|$ exists [**6**] and $|||\mu|||$ is bounded [**3**, Proposition I.1.11]. Consider $A \in \mathcal{F}$, define $\mathcal{F}(A) := \{B \in \mathcal{F} | B \subseteq A\}$, and let $\mathcal{P}(A)$ denote the collection of all finite (disjoint) families $\{A_1, A_2, \ldots, A_n\}$ in $\mathcal{F}(A)$ which satisfy $A = \sum A_i$. Then we have

$$|\mu|(A) = \sup_{\mathcal{F}(A)} (\mu(B) - \mu(A \setminus B)),$$

and we also have

$$||\mu(B) - \mu(A \setminus B)|| \leq |||\mu|||(A)$$

for all $B \in \mathcal{F}(A)$. This yields

$$|| \, |\mu|(A)|| = \left\| \sup_{\mathcal{F}(A)} (\mu(B) - \mu(A \setminus B)) \right\| \leq |||\mu|||(A).$$

Furthermore, we have

$$\left\| \sum \alpha_i \mu(A_i) \right\| \leq \left\| \sum |\alpha_i| \, |\mu(A_i)| \right\| \leq \left\| \sum |\mu|(A_i) \right\| \leq || \, |\mu|(A)||$$

for all $\{A_1, A_2, \ldots, A_n\} \in \mathcal{P}(A)$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in [-1, 1]$. This yields

$$|||\mu|||(A) \leq || \, |\mu|(A)||.$$

Therefore, we have

$$|||\mu|||(A) = || \, |\mu|(A)||,$$

and replacing μ by $|\mu|$ we obtain $||| \, |\mu| \, |||(A) = || \, |\mu|(A)||$. □

The following result is the counterpart of Theorem 2.3 for vector measures:

3.2. THEOREM. *If* $\mathbf{G}$ *is an order complete AM-space with unit, then each strongly additive vector measure* $\mu\colon \mathcal{F} \to \mathbf{G}$ *has a strongly additive modulus* $|\mu|$.

PROOF. By [**3**, Corollary I.1.19], μ is bounded and hence has a bounded modulus $|\mu|$, by Lemma 3.1. By [**3**, Corollary I.1.18], a vector measure $\varphi\colon \mathcal{F} \to \mathbf{G}$ is strongly additive if and only if $\lim |||\varphi|||(A_n) = 0$ holds for each sequence $\{A_n | n \in \mathbf{N}\}$ of mutually disjoint sets in $\mathcal{F}$. It now follows from Lemma 3.1 that $|\mu|$ is strongly additive. □

If $\mathbf{G}$ is order complete, then the ordered vector space $oba(\mathcal{F}, \mathbf{G})$ of all order bounded vector measures $\mathcal{F} \to \mathbf{G}$ is an order complete vector lattice [**6**]. Let $sa(\mathcal{F}, \mathbf{G})$ denote the ordered vector space of all strongly additive vector measures $\mathcal{F} \to \mathbf{G}$.

3.3. COROLLARY. *If* $\mathbf{G}$ *is an order complete AM-space with unit, then* $sa(\mathcal{F}, \mathbf{G})$ *is an ideal in* $oba(\mathcal{F}, \mathbf{G})$.

This is immediate from Theorem 3.2 and Lemma 3.1.

4. Remarks. Lemma 2.1 has an obvious counterpart for vector measures.

Lemma 2.2 provides a general condition on **G** under which each linear operator **E** → **G** having an order bounded adjoint is order bounded. This condition (called *property* (P) in [**4**, **5**, **6**]) fails for c_0 [**2**, p. 64], but it is satisfied for order complete AM-spaces with unit, dual Banach lattices, and KB-spaces [**4**, pp. 251 and 299]. By Schlotterbeck's theorem [**4**, Theorem IV.4.3], the condition on **G** guarantees the lattice property of the ordered vector spaces of all cone absolutely summing operators **E** → **G** and of all majorizing operators **E** → **G**, and it also yields the lattice property of the ordered vector space of all vector measures $\mathcal{F}$ → **G** having bounded variation [**5**, **6**].

Further results on the modulus of order bounded weakly compact operators are given in [**2**, p. 305].

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