PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 102, Number 4, April 1988

OPERATOR THEORETIC CHARACTERIZATIONS OF $[IN]$-GROUPS AND INNER AMENABILITY

ANTHONY TO-MING LAU AND ALAN L. T. PATERSON

ABSTRACT. Let G be a locally compact group and $p \in [1, \infty]$. Let π_p be the isometric representation of G on $L_p(G)$ given by $\pi_p(x)f(t) = f(x^{-1}tx)\Delta(x)^{1/p}$. Let $\mathcal{A}'_p$ be the commutant of $\mathcal{A}_p$ in $B(L_p(G))$. In this paper we determine those G for which: (*) $\mathcal{A}'_p$ contains a nonzero compact operator. We prove, among other things, that if $p \in [1, \infty)$, then (*) holds if and only if $G \in [IN]$, and that if $p = \infty$, then (*) holds if and only if G is inner amenable.

1. Introduction. The left regular representation π_l of a locally compact group G is, of course, of fundamental importance in representation theory. There is, however, another natural representation π_2 of G on $L_2(G)$ associated with conjugation rather than with left translation: here,

$$\pi_2(x)f(t) = f(x^{-1}tx)\Delta(x)^{1/2}$$

($x, t \in G$, $f \in L_2(G)$, and Δ is the modular function of G).

The representation π_2 is related to the existence of inner invariant means on $L_\infty(G)$. A mean m on $L_\infty(G)$ (i.e., $m \in L_\infty(G)^*$; $\|m\| = m(1) = 1$) is *inner invariant* if $m(\phi \circ I_x) = m(\phi)$ for all $\phi \in L_\infty(G)$, $x \in G$, where $I_x: G \to G$ is given by $I_x(y) = xyx^{-1}$. The group G is called *inner amenable* if $L_\infty(G)$ admits an inner invariant mean. The literature on inner amenability has grown substantially in recent years (see [**2, 3, 4, 8, 14, 15, 18, 19**]). It is shown, for example (see [**2, 18, 23**]), that $L_\infty(G)$ has an inner invariant mean which is not the evaluation at the identity e on continuous functions if and only if there exists a state α on $\mathcal{B}(L_2(G))$ such that $\alpha(\pi_2(x)) = 1$ for all $x \in G$.

Let $\mathcal{A}_2$ be the von Neumann algebra generated by $\pi_2(G)$. A natural question is: *When does $\mathcal{A}_2$ contain a nonzero compact operator*? The answer to the corresponding problem for π_l [**5, 13**] is: *When G is compact.* Consideration of the abelian case shows that this is not true for π_2; roughly, $\mathcal{A}_2$ can be too "small." Instead, we formulate the above question for the commutant $\mathcal{A}'_2$ of $\mathcal{A}_2$ in $\mathcal{B}(L_2(G))$. We will show that $\mathcal{A}'_2$ contains a nonzero compact operator if and only if G is an $[IN]$ group, i.e., G contains a compact neighborhood of the identity e which is invariant under every I_x.

More generally, for $p \in [1, \infty]$, let π_p be the isometric representation of G on $L_p(G)$ given by

$$\pi_p(x)f(t) = f(x^{-1}tx)\Delta^{1/p}(x) \qquad (x, t \in G,\ f \in L_p(G)).$$

Received by the editors December 31, 1986.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 43A15, 43A22.

Key words and phrases. Locally compact groups, $[IN]$-groups, fixed-points, operator algebras, compact operators, inner invariant means, amenability.

The first author is supported by the NSERC grant A-7679, and the second author is indebted to the Carnegie Trust for the Universities of Scotland for financial help.

Let $\mathcal{A}'_p$ be the commutant $\pi_p(G)'$ of $\pi_p(G)$. Our main theorem, which we state below, is an immediate consequence of Theorems 1 and 2.

MAIN THEOREM. *If $1 \leq p < \infty$, then $\mathcal{A}'_p$ contains a nonzero compact operator if and only if G is an $[IN]$-group. If $p = \infty$, then $\mathcal{A}'_p$ contains a nonzero compact operator if and only if G is inner amenable.*

2. The case $1 \leq p < \infty$. The equivalence (a) $\Leftrightarrow$ (b) in Theorem 1 was proved for $p = 1$ in [**16**].

THEOREM 1. *Let $1 \leq p < \infty$. The following statements are equivalent:*
(a) *There exists a nonzero $\pi_p(G)$ fixed point in $L_p(G)$.*
(b) *G is an $[IN]$-group.*
(c) *There is a nonzero compact operator in $\mathcal{A}'_p$.*

PROOF. (a)$\Rightarrow$(b). By [**16**], we may assume that $p > 1$. Let $f \in L_p(G)$ be nonzero and $\pi_p(G)$-invariant. Let $q > 1$ be such that $p^{-1} + q^{-1} = 1$ and $h = |f|^{p/q} \in L_q(G)$. Then h is $\pi_q(G)$-fixed and $g = |f| * h^* \in C_0(G)$ by [**10**, 20.16]. Now $\pi_\infty(x)(g) = [\pi_p(x)(|f|)] * [\pi_q(x)(h)]^* = |f| * h^* = g$. So $g(xtx^{-1}) = g(t)$, $x, t \in G$. Now

$$g(e) = \int |f|(y)h^*(y^{-1})\,d\lambda(y) = \|f\|_p^p > 0$$

and so $G \in [IN]$ since $g^{-1}([\frac{1}{2}g(e), \infty))$ is a compact invariant neighborhood of e.

(b) $\Rightarrow$ (c). If G is an $[IN]$-group, then G is unimodular. Let V be a compact invariant neighborhood of e in G. Define $P \in \mathcal{B}(L_p(G))$ by

$$Pf = \int_V f(x)\,d\lambda(x) \cdot 1_V.$$

Then P is a nonzero compact operator in $\mathcal{A}'_p$.

(c) $\Rightarrow$ (a). Let T be a nonzero compact operator in $\mathcal{A}'_p$ and $K = \overline{T(U)}$, where U is the closed unit ball of $L_p(G)$. Then $K \neq \{0\}$ and is convex and norm compact. Furthermore, since $T \in \mathcal{A}'_p$ and each $\pi_p(x)$ is isometric, we see that K is $\pi_p(G)$-invariant. Since $\|\,|f| - |g|\,\|_p \leq \|f - g\|_p$ for $f, g \in L_p(G)$, the set $|K| = \{|f| : f \in K\}$ is norm compact and invariant. Let $f_0 \in |K|$ and $f_0 \neq 0$. If $p = 1$, let K_0 be the closed convex hull of $\{\pi_1(x)f_0;\ x \in G\}$. Then K_0 is compact, convex, invariant and $\|f\|_1 = \|f_0\|_1 > 0$ for each $f \in K$. Hence an application of the Kakutani fixed point theorem shows that there exists a nonzero $\pi_1(G)$ fixed point in K_0.

If $p > 1$, let $1 < q < \infty$ be such that $p^{-1} + q^{-1} = 1$, and $h \in L_q(G)$, $h \geq 0$, such that $\langle f_0, h\rangle > 0$. Let C be the norm closed convex hull of $\pi_q(G)h$ in $L_q(G)$. Since $L_q(G)$ is reflexive, C is weakly compact in $L_q(G)$. Let

$$D = \{hk; h \in C,\ k \in K\} \quad \text{(pointwise product)}.$$

Then D is $\pi_1(G)$-invariant, and a straightforward argument shows that D is weakly compact in $L_1(G)$. By the Krein-Smulian theorem (see [**1**, p. 90]), the norm closed convex hull Φ of D is also weakly compact in $L_1(G)$, and Φ is $\pi_1(G)$-invariant. Let $\Phi_0 = \{\phi \in \Phi; \int \phi(x)\,d\lambda(x) \geq \frac{1}{2}\int f_0(x)h(x)\,d\lambda(x)\}$. Then, clearly, Φ_0 is also weakly compact, convex, $\pi_1(G)$-invariant, and contains f_0h but not 0. An application of the Ryll-Nardzewski (see [**1**, p. 93]) fixed-point theorem to the action of $\pi_1(G)$ on Φ_0 gives a nonzero $\pi_1(G)$-fixed point f in $L_1(G)$. In particular, $|f|^{1/p}$ is a nonzero $\pi_p(G)$-fixed point in $L_p(G)$. $\square$

3. The case $p = \infty$. We will require the following lemma which generalizes the well-known result that the space of weakly almost periodic functions on G admits a G-invariant mean.

Let (X, Σ, μ) be a measure space and $L_\infty(X) = L_\infty(X, \Sigma, \mu)$. Suppose that $L_\infty(X)$ is a right Banach G-module with action $\phi \to \phi x$, such that $1 \cdot x = 1$ and $\|\phi x\| = \|\phi\|$ for all $\phi \in L_\infty(X)$, $x \in G$. Let A be the set of all $\phi \in L_\infty(X)$ such that $\{\phi x;\ x \in G\}$ is relatively compact in the weak-topology of $L_\infty(X)$. Then as in [**1**, Lemma 1.6], A is an invariant closed subspace of $L_\infty(X)$ containing 1.

LEMMA. *There exists a G-invariant mean on A.*

PROOF. Clearly, G acts affinely on the set M of means on A. For $\phi \in A$, let p_ϕ be the seminorm on A^* defined by $p_\phi(m) = \sup\{|m(\phi x)|;\ x \in G\}$. Let τ be the locally convex topology on A^* determined by $\{p_\phi;\ \phi \in A\}$. Then τ lies between the weak topology $\sigma(A^*, A)$ and the Mackey topology for the pair (A^*, A). An application of the Mackey-Arens theorem shows that A is also the continuous dual of (A^*, τ). It follows that M is weakly compact and G acts continuously and distally on (M, τ). By the Ryll-Nardzewski fixed point theorem, there must exist a G-invariant mean on A.

THEOREM 2. *The locally compact group G is inner amenable if and only if there exists a nonzero compact operator in $\mathcal{A}'_\infty$.*

PROOF. If m is an inner invariant mean on $L_\infty(G)$, then the operator S, where $S\phi = m(\phi)1$, is a nonzero compact operator in $\mathcal{A}'_\infty$.

Conversely, suppose that T is a nonzero operator in A'_∞. Let $|T|$ be the modulus of T. So [**22**, Chapter IV, §1] for $\phi \geq 0$ in $L_\infty(G)$, $|T|(\phi) = \sup\{|T\psi|: \psi \in L_\infty(G), |\psi| \leq \phi\}$. By [**22**, Example 3, p. 232], $|T|$ is a compact linear operator. We claim that $|T| \in \mathcal{A}'_\infty$. Indeed, if A is a bounded set in $L_\infty(G)^+$, then $\sup \pi_\infty(x)A = \pi_\infty(x) \sup A$ and so if $\phi \geq 0$,

$$|T|(\pi_\infty(x)\phi) = \sup\{|T(\pi_\infty(x)\psi)| : |\psi| \leq \phi\} = \pi_\infty(x)|T|(\phi).$$

Since $T \neq 0$, $|T| \neq 0$. If $\phi \geq 0$, then $|T(\phi)| \leq \|\phi\|\,|T|(1)$, and it follows that $|T|(1) > 0$. Let $f = |T|(1)$. Since $|T| \in \mathcal{A}'_\infty$ and $\pi_\infty(x)1 = 1$ for all $x \in G$, we have $\pi_\infty(x)f = f$ for all $x \in G$. Let $\varepsilon > 0$ be such that $X = \{x \in G: f(x) \geq \varepsilon\}$ is not locally null. Then $L_\infty(X)$ becomes a right Banach G-module under the action $\phi \to \phi x$, where $\phi x(t) = \phi(xtx^{-1})$ $(t \in X)$. Let $P: L_\infty(G) \to L_\infty(X)$ be the canonical restriction map and regard $L_\infty(X) \subset L_\infty(G)$. Then $P \in A'_\infty$ and so $S = P|T|$ is a compact operator in $\mathcal{A}'_\infty$. For each $\phi \in L_\infty(G)$, $S\phi$ is "almost periodic" in $L_\infty(X)$, and so (in the notation of the lemma) belongs to A. Let m be a G-invariant mean on A. Then $m \circ S$ is an inner invariant, positive linear functional on $L_\infty(G)$. Further, $m \circ S(1) \geq \varepsilon$ so that $m \circ S \neq 0$. Scaling up $m \circ S$ yields an inner invariant mean on $L_\infty(G)$. □

COROLLARY. *If G is connected, then the following statements are equivalent:*

(a) *G is amenable.*

(b) *There is a nonzero compact operator T in $\mathcal{A}'_\infty$.*

PROOF. In this case, as shown by Losert and Rindler [**15**], inner amenability of G is equivalent to amenability. □

4. Notes. (a) *Results related to Theorem* 1. The following two results can be proved by straightforward modifications of the proof of Theorem 1.

THEOREM 3 (CF. [**20**, THEOREM 1]). *The following are equivalent on G:*

(a) *G is an $[IN]$-group.*

(b) *There is a nonzero weakly compact operator in $\mathcal{A}_1'$.*

THEOREM 4. *The following are equivalent on G:*

(a) *G is an $[IN]$-group.*

(b) *There exists a compact convex π_p-invariant subset K in $L_p(G)$, $1 \le p < \infty$, and $K \neq \{0\}$.*

(c) *There exists a weakly compact convex π_1-invariant subset in $L_1(G)$ and $K \neq \{0\}$.*

(b) *Weakly compact operators in $\mathcal{A}_\infty'$.*

THEOREM 5. *The following are equivalent on G:*

(a) *G is inner amenable.*

(b) *There is a weakly compact operator T in $\mathcal{A}_\infty'$ such that $T(h) = 1$ for some $h \in L_\infty(G)$.*

To prove (b)$\Rightarrow$(a) in the above theorem, observe that $\phi \to T\phi$ takes $L_\infty(G)$ into the space A of the lemma of §2 with $X = G$ and conjugation action. If m is a G-invariant mean on A, then $m \circ T$ is invariant and nonzero on $L_\infty(G)$, and so $|m \circ T|/|m \circ T|(1)$ is an inner invariant mean on $L_\infty(G)$ (cf. [**6**, Lemma 2.2]).

(c) *The algebras $\mathcal{A}_2$ and $\mathcal{A}_2'$.* Suppose that G is an $[IN]$-group and $p = 2$. Let $\mathcal{H}_f$ be the fixed-point space $\{f \in L_2(G): \pi(G)f = f\}$, $P: L_2(G) \to \mathcal{H}_f$ be the orthogonal projection and $\mathcal{K} = \mathcal{H}_f^\perp$. Then $P \in \mathcal{A}_2'$. Suppose also that G is amenable. Then the mean ergodic theorem for amenable groups [**9**] shows that $P \in \mathcal{A}_2$. It follows that if $\mathcal{L}$ is the von Neumann algebra generated by $\{\pi_2(x)|_\mathcal{K}: x \in G\}$, then $\mathcal{A}_2 = \mathbf{C}I \oplus \mathcal{L}$, $\mathcal{A}_2' = B(\mathcal{H}_f) \oplus \mathcal{L}'$ on $\mathcal{H}_f \oplus \mathcal{K}$. So the study of $\mathcal{A}_2$ and $\mathcal{A}_2'$ reduces to that of the restriction of π_2 to $\mathcal{K}$.

To illustrate this, take $G = S_3$, the symmetric group on 1, 2, 3. Then [**11**, (27.59)] $\hat{G}$ consists of the three elements 1, ω and σ. Here, ω is the character of G assigning to each $A \in G$ its parity, while σ is the two-dimensional representation on $L = \{\alpha \in \mathbf{C}^3: \alpha_1 + \alpha_2 + \alpha_3 = 0\}$ where $\sigma(A)\alpha = (\alpha_{A(1)}, \alpha_{A(2)}, \alpha_{A(3)})$. One readily checks that $\pi_2 = (31) \oplus \omega \oplus \sigma$. The corresponding Hilbert space decomposition is $\mathcal{H}_f \oplus \mathcal{H}_\omega \oplus \mathcal{H}_\sigma$ where $\mathcal{H}_\sigma = \operatorname{span}\{e, (12) + (23) + (31), (123) + (132)\}$, $\mathcal{H}_w = \mathbf{C}((123) - (132))$ and $\mathcal{H}_\sigma = \operatorname{span}\{(12) - (23), (12) - (31)\}$. So $\mathcal{A}_2 = \mathbf{C}I_3 \oplus \mathbf{C}I_1 \oplus M_2$, while $\mathcal{A}_2' = M_3 \oplus \mathbf{C}I_1 \oplus \mathbf{C}I_2$ (where I_r is the $r \times r$ identity matrix and M_r the algebra of $r \times r$ complex matrices).

(d) *Some open questions.*

(i) The group G is said to be $[SIN]$ if there exists a neighborhood basis for e consisting of compact, invariant sets. *Is there a characterization of $[SIN]$ groups analogous to that for $[IN]$ groups given in Theorem* 1?

(ii) It is well known [**7**, (13.10.5)] that a unimodular G is $[SIN]$ if and only if $VN(G)$ (or $VN(G')$) is finite. *What can be said about $\mathcal{A}_2'$ when G is $[SIN]$*?

(iii) *Is Theorem 2 true if "compact" is replaced by "weakly compact" in its statement*? (Cf. Theorem 4.)

REFERENCES

1. R. B. Burkel, *Weakly almost periodic functions on semigroups*, Gordon and Breach, New York, 1970.
2. H. Choda and M. Choda, *Fullness, simplicity and inner amenability*, Math. Japon. **24** (1979), 235–246.
3. M. Choda, *The factors of inner amenable groups*, Math. Japon. **24** (1979), 145–152.
4. ——, *Effect of inner amenability on strong ergodicity*, Math. Japon. **28** (1983), 109–115.
5. C. Chou, A. T. M. Lau and J. Rosenblatt, *Approximation of compact operators by sums of translations*, Illinois J. Math. **29** (1985), 340–350.
6. M. M. Day, *Semigroups and amenability*, Semigroups (K. W. Folley, ed.), Academic Press, New York, 1969, pp. 5–53.
7. J. Dixmier, *C^*-algebras*, North-Holland, Amsterdam, 1977.
8. E. G. Effros, *Property Γ and inner amenability*, Proc. Amer. Math. Soc. **47** (1975), 483–486.
9. F. P. Greenleaf, *Ergodic theorems and the construction of summing sequences in amenable locally compact groups*, Comm. Pure Appl. Math. **26** (1973), 29–46.
10. E. Hewitt and K. A. Ross, *Abstract harmonic analysis.* I, Springer-Verlag, Berlin and New York, 1963.
11. ——, *Abstract harmonic analysis.* II, Springer-Verlag, Berlin and New York, 1970.
12. G. Köthe, *Topological vector spaces.* I, Springer-Verlag, Berlin and New York, 1969.
13. A. T. M. Lau, *Closed convex invariant subsets of $L_p(G)$*, Trans. Amer. Math. Soc. **232** (1977), 131–142.
14. V. Losert and H. Rindler, *Asymptotically central functions and invariant extensions of Dirac measures*, Probability Measures on Groups, Proc. Conf., Oberwolfach, 1983.
15. ——, *Conjugation invariant means*, Colloq. Math. **51** (1987), 221–225.
16. R. D. Mosak, *Central functions in group algebras*, Proc. Amer. Math. Soc. **29** (1971), 613–616.
17. T. W. Palmer, *Classes of nonabelian, noncompact, locally compact groups*, Rocky Mountain J. Math. **8** (1978), 683–741.
18. W. L . Paschke, *Inner amenability and conjugation operators*, Proc. Amer. Math. Soc. **71** (1978), 117–118.
19. J.-P. Pier, *Quasi-invariance intérieure sur les groupes localement compacts*, G.M.E.L. Actualités Mathématiques, Actes du 6^e Congrès du Groupement des Mathematicians d' Expression Latine, Luxembourg, 1982, pp. 431–436.
20. S. Sakai, *Weakly compact operators on operator algebras*, Pacific J. Math. **14** (1964), 659–664.
21. ——, *C^*-algebras and W^*-algebras*, Springer-Verlag, Berlin and New York, 1971.
22. H. H. Schaefer, *Banach lattices and positive operators*, Springer-Verlag, Berlin and New York, 1974.
23. C. K. Yuan, *The existence of inner invariant means on $L^\infty(G)$*, J. Math. Anal. Appl. (to appear).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALBERTA, EDMONTON, ALBERTA, CANADA T6G 2G1

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ABERDEEN, ABERDEEN, SCOTLAND, UNITED KINGDOM