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A FIXED POINT THEOREM REVISITED

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ABSTRACT. A version of a theorem commonly referred to as Caristi's Theorem is given. It has an elementary constructive proof and it includes many generalizations of Banach's fixed point theorem. Several examples illustrate the diversity that can occur.

THEOREM 1 (CARISTI [1]). *Suppose $T\colon X \to X$ and $\phi\colon X \to [0,\infty)$, where X is a complete metric space and ϕ is lower semicontinuous. If for each x in X,*

$$\text{(C)} \qquad d(x,Tx) \le \phi(x) - \phi(Tx),$$

then T has a fixed point.

THEOREM 2 [2]. *Suppose $T\colon X \to X$, where X is a metric space. Then there exists $\phi\colon X \to [0,\infty)$ such that* (C) *holds if and only if the series*

$$\text{(1)} \qquad \sum_{n=0}^{\infty} d(T^n x, T^{n+1}x)$$

converges for all $x \in X$.

For $x \in X$, $O(x,\infty) = \{x, Tx, T^2x, \dots\}$ is called the orbit of x. $G\colon X \to [0,\infty)$ is T-orbitally lower semicontinuous at x if $\{x_n\}$ is a sequence in $O(x,\infty)$ and $\lim x_n = \bar{x}$ implies $G(\bar{x}) \le \liminf G(x_n)$. If (C) holds for every $y \in O(x,\infty)$, (1) holds for this x, since the sequence of partial sums is nondecreasing and bounded above by $\phi(x)$. This fact is used in the next theorem.

THEOREM 3. *Let (X,d) be a metric space. Suppose $T\colon X \to X$ and $\phi\colon X \to [0,\infty)$. Suppose there exists an x such that*

$$\text{(C)} \qquad d(y,Ty) \le \phi(y) - \phi(Ty)$$

for every $y \in O(x,\infty)$, and any Cauchy sequence in $O(x,\infty)$ converges to a point in X. Then:

(1) *$\lim T^n x = \bar{x}$ exists.*
(2) *$d(T^n x, \bar{x}) \le \phi(T^n x)$.*
(3) *$T\bar{x} = \bar{x}$ iff $G(x) = d(x,Tx)$ is T-orbitally lower semicontinuous at x.*
(4) *$d(T^n x, x) \le \phi(x)$ and $d(\bar{x}, x) \le \phi(x)$.*

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PROOF. Observe that $\{T^n x\}$ is Cauchy. For if $m > n$,

$$\begin{aligned} d(T^n x, T^m x) &\le d(T^n x, T^{n+1} x) + \cdots + d(T^{m-1} x, T^m x) \\ &= \sum_{k=n}^{m-1} d(T^k x, T^{k+1} x). \end{aligned}$$

We noted above that the series $\sum_{k=0}^{\infty} d(T^k x, T^{k+1} x)$ converges. Thus, (1) follows.

$$\begin{aligned} 0 \le d(T^n x, T^m x) &\le \sum_{k=n}^{m-1} d(T^k x, T^{k+1} x) \\ &\le \sum_{k=n}^{m-1} [\phi(T^k x) - \phi(T^{k+1} x)] = \phi(T^n x) - \phi(T^m x) \le \phi(T^n x). \end{aligned}$$

Letting $m \to \infty$ gives (2).

(3) $x_n = T^n x \to \bar{x}$ and G is T-orbitally lower semicontinuous at x implies

$$0 \le d(\bar{x}, T\bar{x}) = G(\bar{x}) \le \liminf G(x_n) = \liminf d(T^n x, T^{n+1} x) = 0.$$

Thus $T\bar{x} = \bar{x}$.

Assume that $T\bar{x} = \bar{x}$ and $\{x_n\}$ is a sequence in $O(x, \infty)$ with $\lim x_n = \bar{x}$. Then $G(\bar{x}) = d(\bar{x}, T\bar{x}) = 0 \le \liminf d(x_n, Tx_n) = \liminf G(x_n)$.

$$\begin{aligned} d(x, T^n x) &\le d(x, Tx) + d(Tx, T^2 x) + \cdots + d(T^{n-1} x, T^n x) \\ &\le [\phi(x) - \phi(Tx)] + [\phi(Tx) - \phi(T^2 x)] + \cdots + [\phi(T^{n-1} x) - \phi(T^n x)] \\ &= \phi(x) - \phi(T^n x) \le \phi(x). \end{aligned} \tag{4}$$

Letting $n \to \infty$ gives $d(x, \bar{x}) \le \phi(x)$.

COROLLARY [**3**]. *Let (X, d) be a complete metric space and $0 < k < 1$. Suppose $T\colon X \to X$ and there exists an x such that*

$$d(Ty, T^2 y) \le k d(y, Ty) \tag{A}$$

for all $y \in O(x, \infty)$. Then:

(1) *$\lim T^n x = \bar{x}$ exists.*

(2) *$d(T^n x, \bar{x}) \le k^n (1-k)^{-1} d(x, Tx)$.*

(3) *$T\bar{x} = \bar{x}$ if and only if $G(x) = d(x, Tx)$ is T-orbitally lower semicontinuous at x.*

(4) *$d(T^n x, x) \le (1-k)^{-1} d(x, Tx)$ and $d(\bar{x}, x) \le (1-k)^{-1} d(x, Tx)$.*

PROOF. Set $\phi(y) = (1-k)^{-1} d(y, Ty)$ for $y \in O(x, \infty)$. Let $y = T^n x$ in (A). Then

$$d(T^{n+1} x, T^{n+2} x) \le k d(T^n x, T^{n+1} x)$$

and

$$d(T^n x, T^{n+1} x) - k d(T^n x, T^{n+1} x) \le d(T^n x, T^{n+1} x) - d(T^{n+1} x, T^{n+2} x).$$

Thus,

$$d(T^n x, T^{n+1} x) \le (1-k)^{-1} [d(T^n x, T^{n+1} x) - d(T^{n+1} x, T^{n+2} x)]$$

or $d(y, Ty) \le \phi(y) - \phi(Ty)$. (1), (3), and (4) are immediate.

For (2), (A) implies $d(T^n x, T^{n+1}x) \leq k^n d(x, Tx)$. Thus,

$$d(T^n x, \bar{x}) \leq \phi(T^n x) = (1-k)^{-1} d(T^n x, T^{n+1}x) \leq k^n (1-k)^{-1} d(x, Tx).$$

REMARKS. (1) ϕ is not required to be lower semicontinuous, and (C) need only hold on $O(x, \infty)$ for some x. Also, it may be as easy to check the lower semicontinuity of G as it is of ϕ. Even when ϕ is lower semicontinuous and (C) holds for all x, Caristi's theorem does not give $T\bar{x} = \bar{x}$, but $Tx_0 = x_0$ for some x_0 in X.

(2) Furthermore (2) and (4) give useful bounds that reduce to the usual Banach bounds in the Corollary and in Banach's theorem. It is shown in [**3** and **4**] that the Corollary includes many generalizations of Banach's fixed point theorem.

EXAMPLE 1. In this example, $T\bar{x} \neq \bar{x}$. However, ϕ is continuous, T is lower semicontinuous, T is continuous except at one point, and $d(x, Tx) = \phi(x) - \phi(Tx)$.

Let $X = [0,1]$ and $\phi(x) = x$ for $x \in X$. Put $Tx = 0$ if $x \in [0, \frac{1}{2}]$, and $Tx = \frac{1}{2}x + \frac{1}{4}$ if $x \in (\frac{1}{2}, 1]$. If $x \in [0, \frac{1}{2}]$, $d(x, Tx) = d(x, 0) = x$, and $\phi(x) - \phi(Tx) = \phi(x) - 0 = x - 0 = x$. Similarly, if $x \in (\frac{1}{2}, 1]$, $d(x, Tx) = \frac{1}{2}x - \frac{1}{4} = \phi(x) - \phi(Tx)$. Thus $d(x, Tx) = \phi(x) - \phi(Tx)$ for all x. Clearly, 0 is the only fixed point of T. If $x > \frac{1}{2}$, $\lim T^n x = \frac{1}{2} \neq T(\frac{1}{2})$.

EXAMPLE 2. This example shows that even if ϕ and T are both continuous, you may have many fixed points. Let X be the subset of the plane, with the usual metric d, defined by $X = \{(x, y) \colon 0 \leq x, y \leq 1\}$. Let $T(x, y) = (x, 0)$. $T(Tp) = Tp$ for every $p \in X$. $0 = d(Tp, T^2 p) \leq \frac{1}{2} d(p, Tp)$. As in the Corollary, let $\phi(p) = 2d(p, Tp)$ and $d(p, Tp) \leq \phi(p) - \phi(Tp)$.

EXAMPLE 3. In this example, T and ϕ are discontinuous, $\phi(x) = cd(x, Tx)$, ϕ is lower semicontinuous, and $d(x, Tx) \leq \phi(x) - \phi(Tx)$. $T \colon [-1, 1] \to [-1, 1]$ is defined by $Tx = -1$, if $x < 0$ and $Tx = x/4$ if $x \geq 0$. Note that $d(Tx, T^2 x) \leq \frac{1}{4} d(x, Tx)$. As in the Corollary, put $\phi(x) = \frac{4}{3} d(x, Tx)$. If $x < 0$, $\lim T^n x = -1 = T(-1)$. If $x > 0$, $\lim T^n x = 0 = T(0)$. Also, 0 and -1 are the only fixed points of T.

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