PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 102, Number 4, April 1988

MEASURES INVARIANT UNDER LOCAL HOMEOMORPHISMS

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(Communicated by R. Daniel Mauldin)

ABSTRACT. Suppose X is a compact Hausdorff space, and G is a set of *local* homeomorphisms of X; sufficient conditions are given for the existence of a G-invariant Borel probability measure P on X. The result generalizes theorems of Mycielski and Steinlage. The proof is an application of the "Loeb measure" construction from nonstandard analysis.

1. Introduction. Suppose X is a compact Hausdorff space, and G is a set of local homeomorphisms of X: does there exist a G-invariant Borel probability measure P on X? The answer is yes under either of the two extra hypotheses:

(i) [Mycielski] X is metric, and G consists of all local isometries of X.

(ii) [Steinlage] G is a weakly transitive group of autohomeomorphisms of X, and for every disjoint pair K, L of compact subsets of X there is an open set u such that for no $g \in G$ does gu simultaneously intersect both K and L.

Here, a *local homeomorphism* (*isometry*) is a homeomorphism (isometry) from one open subset of X onto another, and an *autohomeomorphism* has range $=$ domain $= X$. P is *G-invariant* provided that, for each $g \in G$ and Borel subset E of domain(g), $PE = PgE$. G is *weakly transitive* provided that $X = \bigcup_{g\in G} gu$ for every open subset u of X.

Steinlage's theorem is a strong generalization of the existence theorem for Haar measure; Mycielski's theorem is a partial solution to the second problem in the *Scottish book* [**2**]. The proofs of these results are difficult, and do not resemble one another.

This paper presents a simple proof of a theorem which simultaneously generalizes the above results. Suppose U is an open cover of X, and $G' \subseteq G$; call U G'-*distributed* provided whenever $u \in U$, $g \in G'$, and $u \subseteq$ domain(g), $gu \in U$. If, in addition, K' is a collection of compact sets, call U K'-*small* provided that whenever $u \in U$, and K and L are disjoint elements of K', then either $u \cap K = \varnothing$ or $u \cap L = \varnothing$.

THEOREM 1. *Let X be a compact Hausdorff space, and let G be a set of local homeomorphisms of X, closed under inverses. Suppose that for every finite $G' \subseteq G$, and finite collection K' of compact sets, there is a G'-distributed K'-small open cover U of X. Then there is a G-invariant Borel probability measure P on X.*

2. Preliminaries. For simplicity, the definitions will be rephrased in the language of nonstandard analysis, and proved by means of Loeb's hyperfinite measure

Received by the editors January 13, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 28A12, 28C10; Secondary 28E05.

[1]The author is supported by an SERC grant.

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0002-9939/88 $1.00 + $.25 per page

construction. The reader is referred to [**1** or **7**] for definitions, notation, and details. Some major facts are summarized here.

Suppose X is compact Hausdorff and $\Omega^* \subseteq X$ is *finite. If $\underline{\underline{A}}$ is an internal subalgebra of ${}^*P(\Omega)$, the algebra of internal subsets of Ω, then $L(\underline{\underline{A}})$ denotes the smallest (external) σ-algebra containing $\underline{\underline{A}}$. Write $L(\Omega) = L(\underline{\underline{A}})$ when $\underline{\underline{A}} = {}^*P(\Omega)$.

If μ is an internal *probability measure on $(\Omega, \underline{\underline{A}})$, then there is a standard probability space $(\Omega, L(\underline{\underline{A}}), L(\mu))$ such that $L(\mu)(E) = \sup\{{}^\circ\mu(A) : A \in \underline{\underline{A}},\, A \subseteq E\}$ for $E \in L(\underline{\underline{A}})$; in particular, $L(\mu)(A) = {}^\circ\mu(A)$ for $A \in \underline{\underline{A}}$. Denote by $L'(\underline{\underline{A}})$ the completion of $L(\underline{\underline{A}})$ under $L(\mu)$.

If E is a Baire subset of X, and Ω is *S-dense* (that is, $X = \{{}^\circ x : x \in \Omega\}$), then $\Omega \cap \mathrm{st}^{-1} E \in L(\Omega)$. Thus, if μ is an internal probability measure on $(\Omega, {}^*P(\Omega))$, then there is a natural image Baire probability measure P on X, given by $P(E) = L(\mu)(\mathrm{st}^{-1} E)$. The completion of this Baire measure, which includes the Borel sets B_X, is the image of $(\Omega, L'(\Omega), L(\mu))$.

If $u \subseteq {}^*X$, then call u an *infinitesimal neighborhood* provided u is *open, and whenever v is an open subset of X with $u \cap {}^*v \neq \varnothing$, $u \subseteq {}^*v$. Assuming the nonstandard model is sufficiently saturated, every $x \in X$ is contained in an infinitesimal neighborhood. If g is a local homeomorphism of X, and u is an infinitesimal neighborhood contained in ${}^*\,\mathrm{domain}(g)$, then ${}^*g(u)$ is another infinitesimal neighborhood.

For A a *finite set, denote by $\|A\|$ the internal cardinality of A. If B is any (standard) set, let ${}^\sigma B = \{{}^*b : b \in B\}$.

3. Proof of main result.

LEMMA 1. *Suppose X is a compact Hausdorff space, and G is a set of local homeomorphisms of X, closed under inverses. The following are equivalent:*

(i) *For every finite $G' \subseteq G$, and finite collection K' of compact sets, there is a G'-distributed K'-small open cover U of X.*

(ii) *There is a ${}^\sigma G$-distributed *open *cover U of ${}^*X$, with each $u \in U$ an infinitesimal neighborhood.*

PROOF. (i) $\to$ (ii). By saturation there is a ${}^\sigma G$-distributed *open *cover of ${}^*X$ such that, whenever K and L are disjoint compact subsets of X and $u \in U$, either $u \cap {}^*K = \varnothing$ or $u \cap {}^*L = \varnothing$. It suffices to show that this last condition ensures that every u in U is infinitesimal. Let $x, y \in u$, and suppose ${}^\circ x \neq {}^\circ y$. There is an open set v containing ${}^\circ x$ and an open set w containing ${}^\circ y$, such that the closures K of v and L of w are disjoint. However, $x \in u \cap \mathrm{st}^{-1}\,({}^\circ x) \subseteq u \cap {}^*K$ and $y \in u \cap \mathrm{st}^{-1}({}^\circ y) \subseteq u \cap {}^*L$, a contradiction. Thus ${}^\circ x = {}^\circ y$, so $x \approx y$ and u is infinitesimal.

(ii) $\to$ (i). Fix G' and K', and let U be the *open cover from (ii). If $g \in G'$, then U is $\{{}^*g\}$-distributed so (since G' is finite) U is ${}^*G'$-distributed. If K and L are disjoint elements of K', and $u \in U$, the Hausdorff hypothesis on X guarantees that either $u \cap {}^*K = \varnothing$ or $u \cap {}^*L = \varnothing$; from this and finiteness of K' it follows that U is ${}^*K'$-small. By transfer there is a G'-distributed K'-small open cover of X. □

PROOF OF THEOREM 1. By Lemma 1 let U be a ${}^\sigma G$-distributed *open *cover of ${}^*X$. Let $\{u_1, \ldots, u_H\}$ be a *finite subcover of ${}^*X$ from U; such a subcover exists since U is internal and ${}^*X$ is *compact. Choose this subcover so that H is *minimum.

For $i \leq H$, choose $x_i \in U_i$; this can be done in such a way that $\Omega = \{x_1, \ldots, x_H\}$ is internal, and $x_i \neq x_j$ whenever $i \neq j$. There is an internal function $u\colon {}^*X \to U$ such that $x \in u(x)$ for all $x \in {}^*X$, and $u(x_i) = u_i$ for every $i \leq H$. Evidently Ω is S-dense.

Let $\underline{\underline{A}} = {}^*P(\Omega)$, and for $A \in \underline{\underline{A}}$ put $\mu(A) = \|A\|/H$. Extend the internal *probability $(\Omega, \underline{\underline{A}}, \mu)$ to the complete probability space $(\Omega, L'(\Omega), L(\mu))$ by means of the Loeb construction. Let (X, B_X, P) be the image under the standard part map of this measure. It remains to show that P is G-invariant.

Take $g \in G$, $E \in B_X$ contained in the domain of g, and let A be any internal subset of $\Omega \cap \operatorname{st}^{-1} E$. It suffices to produce an internal subset B of $\Omega \cap \operatorname{st}^{-1} g(E)$ with $\|B\| \geq \|A\|$ (since then $P(E) \leq P(g(E))$; the same argument applied to g^{-1} proves $P(E) = P(g(E))$).

For $i \leq H$ let $i^+ = \{j \leq H\colon {}^*g(u_i) \cap u_j \neq \varnothing\}$. Put $B = \{x_j : j \in i^+ \text{ for some } x_i \in A\}$. Since each u_i is an infinitesimal neighborhood of x_i whenever $x_i \in A$ and $j \in i^+$ we have $u_i \subseteq \operatorname{st}^{-1} E$ and $u_j \subseteq \operatorname{st}^{-1} g(E)$. Thus $B \subseteq \operatorname{st}^{-1} g(E)$.

Observe that $\{u_i : x_i \in (\Omega \backslash A)\} \cup \{u(x) : x \in g^{-1}(B)\}$ is a subcover from U. Since H was chosen to be a minimum, $\|B\| \geq \|A\|$. The theorem is proved. □

REMARK. The last paragraph of the above proof actually shows that for every internal $D \subseteq \{1, \ldots, H\}$, $\|\bigcup_{i \in D} i^+\| \geq \|D\|$. Since the map $i \to i^+$ is internal, an internal application of Hall's "Marriage Theorem" produces an internal permutation π of $\{1, \ldots, H\}$ with $\pi(i) \in i^+$ for all i. It follows that there is a map θ from G into the *group S_H of internal permutations of $\{1, \ldots, H\}$, with the property that if $g \in G$ and $\pi = \theta(g)$ then $g(x_i) \approx x_{\pi(i)}$ for all i. With care, this θ can be made an isomorphism.

4. Applications.

COROLLARY 1 (MYCIELSKI'S THEOREM). *Suppose X is compact metric, and G is the set of local isometries of X. Let ε be positive infinitesimal, and U the set of all ε-balls in ${}^*X$. If $g \in G$ has domain v, and $u \in U$ is a subset of ${}^*v$, then ${}^*g(u)$ is an ε-ball because ${}^*g$ is an *isometry. U is therefore ${}^\sigma G$-distributed, and Mycielski's theorem follows from Theorem* 1. □

COROLLARY 2 (STEINLAGE'S THEOREM). *Suppose the hypotheses of Steinlage's theorem are satisfied. Fix $G' \subseteq G$ finite, and let K' be a finite collection of compact subsets of X. For every pair $\gamma = \{K, L\}$ of disjoint elements of K' there is, by hypothesis, an open set u_γ no image of which simultaneously intersects both K and L. By weak transitivity of G, $U_\gamma = \{gu_\gamma : g \in G\}$ covers X.*

Let U consist of all open sets of the form $\bigcap V$, where V contains exactly one element from each U_γ. Since each U_γ is a G-distributed open cover of X, and K' is finite, U is a G-distributed open cover of X. Clearly U is K'-small. The hypotheses of Theorem 1 *are satisfied, so there is a G-invariant Borel probability measure on X.* □

COROLLARY 3 (HAAR MEASURE). *Suppose X is a compact topological group (see* [**4**] *for definitions). For any $z \in X$, let g_z be the function $x \to xz$. Then $G = \{g_z\colon z \in X\}$ is a group of homeomorphisms of X. Haar measure is a G-invariant Borel probability measure on X.*

Fix any infinitesimal neighborhood u in $^*X$, and let $U = \{g(u) \colon g \in {}^*G\}$. U is clearly a $^\sigma G$-distributed cover of $^*X$. The existence of Haar measure will follow from Theorem 1 once it is proved that every element of U is an infinitesimal neighborhood.

Suppose that $g = g_z \in {}^*G$, $z \in {}^*X$. Since g is a *homeomorphism, $g(u)$ is *open. Take any $x, y \in u$. Note that

$$g(x) = xz = x^\circ z(^\circ z)^{-1}z \approx x^\circ zz^{-1}z = x^\circ z,$$

where continuity of the function $z \to z^{-1}$ implies that $(^\circ z)^{-1} \approx z^{-1}$. Similarly, $g(y) \approx y^\circ z$. Since $x \approx y$, $g(x) \approx g(y)$, so $g(u)$ is an infinitesimal neighborhood. □

For the next application, recall that a *uniformity* on X is a collection $T = \{U_\alpha\}$ of open covers of X such that each pair U_α, U_β has a common barycentric refinement U_γ. That is, for every $x \in X$ there are sets $u \in U_\alpha$ and $v \in U_\beta$ with $U_\gamma[x] \subseteq (u \cap v)$, where $U_\gamma[x] = \bigcup\{w \in U_\gamma : x \in w\}$. The uniformity T is *compatible* with X provided the sets $U[x]$, where $x \in X$ and $U \in T$, form a base for the topology of X. Call a local homeomorphism f of X *T-preserving* provided $f(u) \in U$ whenever $u \in U \in T$ and u is contained in the domain of f.

COROLLARY 4. *Suppose X is compact Hausdorff, $T = \{U_\alpha\}$ is a uniformity compatible with X, and G is the set of T-preserving local homeomorphisms of X. By saturation there is a $U \in {}^*T$ which *refines each standard U_α. By definition of G, U in $^*G$-distributed, and it is easy to see that each $u \in U$ is infinitesimal, so X has a G-invariant Borel probability measure.* □

In the examples discussed so far, X has a local uniform structure which G preserves. The last application is a space X with a natural local structure *not* preserved by G.

EXAMPLE 1. Let (Y, d) and (T, ρ) be metric spaces, with T compact. Fix any group H of autoisometries of Y with the property that if H' is a finite subset of H, and K is a compact subset of Y, then $\{h^n(K) \colon h \in H', n \in Z\}$ has compact closure. (Examples include rotations around the origin of $\mathbf{R}^2$, and finite permutations on a discrete space Y.) Let Γ be the set of components of Y, and for $\gamma \in \Gamma$ denote by I_γ the characteristic function of γ.

Let X be the space of 1-*Lipschitz* functions from Y to T; that is, $f \in X$ provided $\rho(f(y), f(z)) \leq d(y, z)$ for all $y, z \in Y$. Ascoli's theorem ensures that this space is compact in the topology of uniform convergence on compact sets. Recall that a basic open set for this topology has the form $N(f, K, \varepsilon) = \{f' \in X : \rho(f(y), f'(y)) < \varepsilon$ for all $y \in K\}$, where $f \in X$, K is a compact subset of Y, and $\varepsilon > 0$.

Let G be the set of all autohomeomorphisms of X of the form $g(x) = x \circ h$, where $h \in H$, together with those of the form $g(x) = \sum_{\gamma \in \Gamma} h_\gamma \circ x \circ I_\gamma$, where for $\gamma \in \Gamma$ h_γ is an autoisometry of T.

By saturation and the definition of H, there is a *compact subset of K of $^*Y$ such that, for every standard compact $K' \subseteq Y$ and every $h \in H$, $^*K' \subseteq K = {}^*h(K)$. Put $\varepsilon > 0$ infinitesimal, and let $U = \{N(f, K, \varepsilon) : f \in {}^*X\}$. Since K is $^\sigma H$-invariant, U is $^\sigma G$-distributed; since K contains all standard compact sets and $\varepsilon \approx 0$, each $u \in U$ is infinitesimal. It follows from Theorem 1 that there is a G-invariant Borel probability measure on X.

Could this measure have been obtained as a consequence of Corollary 1 or 2? Suppose H has the additional property (satisfied, for example, by finite permutations on a discrete Y) that if K is compact then $h(K) \cap K = \varnothing$ for some $h \in H$.

Fix $t \in T$, let τ be the constant function $\tau(y) = t$, and let f be any element of $X - \{\tau\}$ with the property that $f(y) = t$ for all y off some compact subset K of Y.

Suppose u is any open neighborhood of τ; for some $\varepsilon > 0$ and compact $K' \subseteq Y$, $N(\tau, K', \varepsilon) \subseteq u$. Choose $h \in H$ satisfying $h(K \cup K') \cap (K \cup K') = \varnothing$, and let g be the autohomeomorphism $g(x) = x \circ h$. Since $g(\tau) = \tau$, $g(N(\tau, K', \varepsilon)) = N(\tau, h(K'), \varepsilon)$. Thus both τ and f are elements of $g(u)$.

The disjoint pair of compact sets $\{f\}$ and $\{\tau\}$ provide an immediate counterexample to the hypotheses of Steinlage's theorem. As for Mycielski's theorem, observe that X is, in general, not metrizable. Moreover, in those cases where X is metrizable with a metric η, then some element of G takes the δ-ball around τ, where $\delta = \eta(\tau, f)/2$, onto an open set containing both τ and f; it follows that G is not a set of isometries, so the hypotheses of Mycielski's theorem fail as well. □

5. Extensions. In [**6**], Steinlage proves his theorem not just for compact Hausdorff spaces, but indeed for any *locally* compact Hausdorff space X. Theorem 1 extends to such spaces as well, provided the hypothesis is strengthened and the conclusion weakened as follows: In the hypothesis replace the word "compact" by the word "closed"; in the conclusion, replace "Borel" by "Baire". The proof is quite the same, though technically more involved. See [**5**] for a discussion of the special case where G is a group of autohomeomorphisms.

The author does not know whether, in the locally compact case, Theorem 1 holds with "compact" in place of "closed".

If X is a metric space and G is the set of local isometries, then Theorem 1 is no stronger than Mycielski's theorem. In particular, it gives no more information about the *Scottish book* problem mentioned in §1. That problem is to find a measure invariant under *all* partial isometries, not just those with open domain and range. Christoph Bandt has shown that such a measure exists provided the space X is locally homogeneous, i.e., for every $x, y \in X$ there is a local isometry g of X with $g(x) = y$.

6. Acknowledgments. The author is grateful to Nigel Cutland for discussion and advice, and to the referee for pointing out a terminological ambiguity in the original manuscript and for describing Bandt's result mentioned above.

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