

## MEASURES INVARIANT UNDER LOCAL HOMEOMORPHISMS

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(Communicated by R. Daniel Mauldin)

**ABSTRACT.** Suppose  $X$  is a compact Hausdorff space, and  $G$  is a set of local homeomorphisms of  $X$ ; sufficient conditions are given for the existence of a  $G$ -invariant Borel probability measure  $P$  on  $X$ . The result generalizes theorems of Mycielski and Steinlage. The proof is an application of the "Loeb measure" construction from nonstandard analysis.

**1. Introduction.** Suppose  $X$  is a compact Hausdorff space, and  $G$  is a set of local homeomorphisms of  $X$ : does there exist a  $G$ -invariant Borel probability measure  $P$  on  $X$ ? The answer is yes under either of the two extra hypotheses:

(i) [Mycielski]  $X$  is metric, and  $G$  consists of all local isometries of  $X$ .

(ii) [Steinlage]  $G$  is a weakly transitive group of autohomeomorphisms of  $X$ , and for every disjoint pair  $K, L$  of compact subsets of  $X$  there is an open set  $u$  such that for no  $g \in G$  does  $gu$  simultaneously intersect both  $K$  and  $L$ .

Here, a *local homeomorphism (isometry)* is a homeomorphism (isometry) from one open subset of  $X$  onto another, and an *autohomeomorphism* has range = domain =  $X$ .  $P$  is  $G$ -invariant provided that, for each  $g \in G$  and Borel subset  $E$  of domain( $g$ ),  $PE = PgE$ .  $G$  is *weakly transitive* provided that  $X = \bigcup_{g \in G} gu$  for every open subset  $u$  of  $X$ .

Steinlage's theorem is a strong generalization of the existence theorem for Haar measure; Mycielski's theorem is a partial solution to the second problem in the *Scottish book* [2]. The proofs of these results are difficult, and do not resemble one another.

This paper presents a simple proof of a theorem which simultaneously generalizes the above results. Suppose  $U$  is an open cover of  $X$ , and  $G' \subseteq G$ ; call  $U$   $G'$ -distributed provided whenever  $u \in U$ ,  $g \in G'$ , and  $u \subseteq \text{domain}(g)$ ,  $gu \in U$ . If, in addition,  $K'$  is a collection of compact sets, call  $U$   $K'$ -small provided that whenever  $u \in U$ , and  $K$  and  $L$  are disjoint elements of  $K'$ , then either  $u \cap K = \emptyset$  or  $u \cap L = \emptyset$ .

**THEOREM 1.** *Let  $X$  be a compact Hausdorff space, and let  $G$  be a set of local homeomorphisms of  $X$ , closed under inverses. Suppose that for every finite  $G' \subseteq G$ , and finite collection  $K'$  of compact sets, there is a  $G'$ -distributed  $K'$ -small open cover  $U$  of  $X$ . Then there is a  $G$ -invariant Borel probability measure  $P$  on  $X$ .*

**2. Preliminaries.** For simplicity, the definitions will be rephrased in the language of nonstandard analysis, and proved by means of Loeb's hyperfinite measure

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Received by the editors January 13, 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 28A12, 28C10; Secondary 28E05.

<sup>1</sup>The author is supported by an SERC grant.

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construction. The reader is referred to [1 or 7] for definitions, notation, and details. Some major facts are summarized here.

Suppose  $X$  is compact Hausdorff and  $\Omega^* \subseteq X$  is  $*$ finite. If  $\underline{A}$  is an internal subalgebra of  $*P(\Omega)$ , the algebra of internal subsets of  $\Omega$ , then  $L(\underline{A})$  denotes the smallest (external)  $\sigma$ -algebra containing  $\underline{A}$ . Write  $L(\Omega) = L(\underline{A})$  when  $\underline{A} = *P(\Omega)$ .

If  $\mu$  is an internal  $*$ probability measure on  $(\Omega, \underline{A})$ , then there is a standard probability space  $(\Omega, L(\underline{A}), L(\mu))$  such that  $L(\mu)(E) = \sup\{\circ\mu(A) : A \in \underline{A}, A \subseteq E\}$  for  $E \in L(\underline{A})$ ; in particular,  $L(\mu)(A) = \circ\mu(A)$  for  $A \in \underline{A}$ . Denote by  $L'(\underline{A})$  the completion of  $L(\underline{A})$  under  $L(\mu)$ .

If  $E$  is a Baire subset of  $X$ , and  $\Omega$  is  $S$ -dense (that is,  $X = \{x : x \in \Omega\}$ ), then  $\Omega \cap \text{st}^{-1} E \in L(\Omega)$ . Thus, if  $\mu$  is an internal probability measure on  $(\Omega, *P(\Omega))$ , then there is a natural image Baire probability measure  $P$  on  $X$ , given by  $P(E) = L(\mu)(\text{st}^{-1} E)$ . The completion of this Baire measure, which includes the Borel sets  $B_X$ , is the image of  $(\Omega, L'(\Omega), L(\mu))$ .

If  $u \subseteq *X$ , then call  $u$  an *infinitesimal neighborhood* provided  $u$  is  $*$ open, and whenever  $v$  is an open subset of  $X$  with  $u \cap *v \neq \emptyset$ ,  $u \subseteq *v$ . Assuming the nonstandard model is sufficiently saturated, every  $x \in X$  is contained in an infinitesimal neighborhood. If  $g$  is a local homeomorphism of  $X$ , and  $u$  is an infinitesimal neighborhood contained in  $*$  domain( $g$ ), then  $*g(u)$  is another infinitesimal neighborhood.

For  $A$  a  $*$ finite set, denote by  $\|A\|$  the internal cardinality of  $A$ . If  $B$  is any (standard) set, let  $\sigma B = \{ *b : b \in B \}$ .

**3. Proof of main result.**

LEMMA 1. *Suppose  $X$  is a compact Hausdorff space, and  $G$  is a set of local homeomorphisms of  $X$ , closed under inverses. The following are equivalent:*

- (i) *For every finite  $G' \subseteq G$ , and finite collection  $K'$  of compact sets, there is a  $G'$ -distributed  $K'$ -small open cover  $U$  of  $X$ .*
- (ii) *There is a  $\sigma G$ -distributed  $*$ open  $*$ cover  $U$  of  $*X$ , with each  $u \in U$  an infinitesimal neighborhood.*

PROOF. (i)  $\rightarrow$  (ii). By saturation there is a  $\sigma G$ -distributed  $*$ open  $*$ cover of  $*X$  such that, whenever  $K$  and  $L$  are disjoint compact subsets of  $X$  and  $u \in U$ , either  $u \cap *K = \emptyset$  or  $u \cap *L = \emptyset$ . It suffices to show that this last condition ensures that every  $u$  in  $U$  is infinitesimal. Let  $x, y \in u$ , and suppose  $\circ x \neq \circ y$ . There is an open set  $v$  containing  $\circ x$  and an open set  $w$  containing  $\circ y$ , such that the closures  $K$  of  $v$  and  $L$  of  $w$  are disjoint. However,  $x \in u \cap \text{st}^{-1}(\circ x) \subseteq u \cap *K$  and  $y \in u \cap \text{st}^{-1}(\circ y) \subseteq u \cap *L$ , a contradiction. Thus  $\circ x = \circ y$ , so  $x \approx y$  and  $u$  is infinitesimal.

(ii)  $\rightarrow$  (i). Fix  $G'$  and  $K'$ , and let  $U$  be the  $*$ open cover from (ii). If  $g \in G'$ , then  $U$  is  $\{ *g \}$ -distributed so (since  $G'$  is finite)  $U$  is  $*G'$ -distributed. If  $K$  and  $L$  are disjoint elements of  $K'$ , and  $u \in U$ , the Hausdorff hypothesis on  $X$  guarantees that either  $u \cap *K = \emptyset$  or  $u \cap *L = \emptyset$ ; from this and finiteness of  $K'$  it follows that  $U$  is  $*K'$ -small. By transfer there is a  $G'$ -distributed  $K'$ -small open cover of  $X$ .  $\square$

PROOF OF THEOREM 1. By Lemma 1 let  $U$  be a  $\sigma G$ -distributed  $*$ open  $*$ cover of  $*X$ . Let  $\{u_1, \dots, u_H\}$  be a  $*$ finite subcover of  $*X$  from  $U$ ; such a subcover exists since  $U$  is internal and  $*X$  is  $*$ compact. Choose this subcover so that  $H$  is  $*$ minimum.

For  $i \leq H$ , choose  $x_i \in U_i$ ; this can be done in such a way that  $\Omega = \{x_1, \dots, x_H\}$  is internal, and  $x_i \neq x_j$  whenever  $i \neq j$ . There is an internal function  $u: {}^*X \rightarrow U$  such that  $x \in u(x)$  for all  $x \in {}^*X$ , and  $u(x_i) = u_i$  for every  $i \leq H$ . Evidently  $\Omega$  is  $S$ -dense.

Let  $\underline{A} = {}^*P(\Omega)$ , and for  $A \in \underline{A}$  put  $\mu(A) = \|A\|/H$ . Extend the internal  ${}^*$ probability  $(\Omega, \underline{A}, \mu)$  to the complete probability space  $(\Omega, L'(\Omega), L(\mu))$  by means of the Loeb construction. Let  $(X, B_X, P)$  be the image under the standard part map of this measure. It remains to show that  $P$  is  $G$ -invariant.

Take  $g \in G$ ,  $E \in B_X$  contained in the domain of  $g$ , and let  $A$  be any internal subset of  $\Omega \cap \text{st}^{-1} E$ . It suffices to produce an internal subset  $B$  of  $\Omega \cap \text{st}^{-1} g(E)$  with  $\|B\| \geq \|A\|$  (since then  $P(E) \leq P(g(E))$ ); the same argument applied to  $g^{-1}$  proves  $P(E) = P(g(E))$ .

For  $i \leq H$  let  $i^+ = \{j \leq H : {}^*g(u_i) \cap u_j \neq \emptyset\}$ . Put  $B = \{x_j : j \in i^+ \text{ for some } x_i \in A\}$ . Since each  $u_i$  is an infinitesimal neighborhood of  $x_i$  whenever  $x_i \in A$  and  $j \in i^+$  we have  $u_i \subseteq \text{st}^{-1} E$  and  $u_j \subseteq \text{st}^{-1} g(E)$ . Thus  $B \subseteq \text{st}^{-1} g(E)$ .

Observe that  $\{u_i : x_i \in (\Omega \setminus A)\} \cup \{u(x) : x \in g^{-1}(B)\}$  is a subcover from  $U$ . Since  $H$  was chosen to be a minimum,  $\|B\| \geq \|A\|$ . The theorem is proved.  $\square$

REMARK. The last paragraph of the above proof actually shows that for every internal  $D \subseteq \{1, \dots, H\}$ ,  $\|\bigcup_{i \in D} i^+\| \geq \|D\|$ . Since the map  $i \rightarrow i^+$  is internal, an internal application of Hall's "Marriage Theorem" produces an internal permutation  $\pi$  of  $\{1, \dots, H\}$  with  $\pi(i) \in i^+$  for all  $i$ . It follows that there is a map  $\theta$  from  $G$  into the  ${}^*$ group  $S_H$  of internal permutations of  $\{1, \dots, H\}$ , with the property that if  $g \in G$  and  $\pi = \theta(g)$  then  $g(x_i) \approx x_{\pi(i)}$  for all  $i$ . With care, this  $\theta$  can be made an isomorphism.

#### 4. Applications.

COROLLARY 1 (MYCIELSKI'S THEOREM). *Suppose  $X$  is compact metric, and  $G$  is the set of local isometries of  $X$ . Let  $\varepsilon$  be positive infinitesimal, and  $U$  the set of all  $\varepsilon$ -balls in  ${}^*X$ . If  $g \in G$  has domain  $v$ , and  $u \in U$  is a subset of  ${}^*v$ , then  ${}^*g(u)$  is an  $\varepsilon$ -ball because  ${}^*g$  is an  ${}^*$ isometry.  $U$  is therefore  ${}^\sigma G$ -distributed, and Mycielski's theorem follows from Theorem 1.  $\square$*

COROLLARY 2 (STEINLAGE'S THEOREM). *Suppose the hypotheses of Steinlage's theorem are satisfied. Fix  $G' \subseteq G$  finite, and let  $K'$  be a finite collection of compact subsets of  $X$ . For every pair  $\gamma = \{K, L\}$  of disjoint elements of  $K'$  there is, by hypothesis, an open set  $u_\gamma$  no image of which simultaneously intersects both  $K$  and  $L$ . By weak transitivity of  $G$ ,  $U_\gamma = \{gu_\gamma : g \in G'\}$  covers  $X$ .*

*Let  $U$  consist of all open sets of the form  $\bigcap V$ , where  $V$  contains exactly one element from each  $U_\gamma$ . Since each  $U_\gamma$  is a  $G$ -distributed open cover of  $X$ , and  $K'$  is finite,  $U$  is a  $G$ -distributed open cover of  $X$ . Clearly  $U$  is  $K'$ -small. The hypotheses of Theorem 1 are satisfied, so there is a  $G$ -invariant Borel probability measure on  $X$ .  $\square$*

COROLLARY 3 (HAAR MEASURE). *Suppose  $X$  is a compact topological group (see [4] for definitions). For any  $z \in X$ , let  $g_z$  be the function  $x \rightarrow xz$ . Then  $G = \{g_z : z \in X\}$  is a group of homeomorphisms of  $X$ . Haar measure is a  $G$ -invariant Borel probability measure on  $X$ .*

Fix any infinitesimal neighborhood  $u$  in  $*X$ , and let  $U = \{g(u) : g \in *G\}$ .  $U$  is clearly a  ${}^\sigma G$ -distributed cover of  $*X$ . The existence of Haar measure will follow from Theorem 1 once it is proved that every element of  $U$  is an infinitesimal neighborhood.

Suppose that  $g = g_z \in *G$ ,  $z \in *X$ . Since  $g$  is a  $*$ homeomorphism,  $g(u)$  is  $*$ open. Take any  $x, y \in u$ . Note that

$$g(x) = xz = x^\circ z({}^\circ z)^{-1} z \approx x^\circ z z^{-1} z = x^\circ z,$$

where continuity of the function  $z \rightarrow z^{-1}$  implies that  $({}^\circ z)^{-1} \approx z^{-1}$ . Similarly,  $g(y) \approx y^\circ z$ . Since  $x \approx y$ ,  $g(x) \approx g(y)$ , so  $g(u)$  is an infinitesimal neighborhood.  $\square$

For the next application, recall that a *uniformity* on  $X$  is a collection  $T = \{U_\alpha\}$  of open covers of  $X$  such that each pair  $U_\alpha, U_\beta$  has a common barycentric refinement  $U_\gamma$ . That is, for every  $x \in X$  there are sets  $u \in U_\alpha$  and  $v \in U_\beta$  with  $U_\gamma[x] \subseteq (u \cap v)$ , where  $U_\gamma[x] = \bigcup \{w \in U_\gamma : x \in w\}$ . The uniformity  $T$  is *compatible* with  $X$  provided the sets  $U[x]$ , where  $x \in X$  and  $U \in T$ , form a base for the topology of  $X$ . Call a local homeomorphism  $f$  of  $X$  *T-preserving* provided  $f(u) \in U$  whenever  $u \in U \in T$  and  $u$  is contained in the domain of  $f$ .

**COROLLARY 4.** *Suppose  $X$  is compact Hausdorff,  $T = \{U_\alpha\}$  is a uniformity compatible with  $X$ , and  $G$  is the set of  $T$ -preserving local homeomorphisms of  $X$ . By saturation there is a  $U \in *T$  which  $*$ refines each standard  $U_\alpha$ . By definition of  $G$ ,  $U$  is  ${}^\sigma G$ -distributed, and it is easy to see that each  $u \in U$  is infinitesimal, so  $X$  has a  $G$ -invariant Borel probability measure.  $\square$*

In the examples discussed so far,  $X$  has a local uniform structure which  $G$  preserves. The last application is a space  $X$  with a natural local structure *not* preserved by  $G$ .

**EXAMPLE 1.** Let  $(Y, d)$  and  $(T, \rho)$  be metric spaces, with  $T$  compact. Fix any group  $H$  of autoisometries of  $Y$  with the property that if  $H'$  is a finite subset of  $H$ , and  $K$  is a compact subset of  $Y$ , then  $\{h^n(K) : h \in H', n \in \mathbb{Z}\}$  has compact closure. (Examples include rotations around the origin of  $\mathbb{R}^2$ , and finite permutations on a discrete space  $Y$ .) Let  $\Gamma$  be the set of components of  $Y$ , and for  $\gamma \in \Gamma$  denote by  $I_\gamma$  the characteristic function of  $\gamma$ .

Let  $X$  be the space of 1-Lipschitz functions from  $Y$  to  $T$ ; that is,  $f \in X$  provided  $\rho(f(y), f(z)) \leq d(y, z)$  for all  $y, z \in Y$ . Ascoli's theorem ensures that this space is compact in the topology of uniform convergence on compact sets. Recall that a basic open set for this topology has the form  $N(f, K, \varepsilon) = \{f' \in X : \rho(f(y), f'(y)) < \varepsilon \text{ for all } y \in K\}$ , where  $f \in X$ ,  $K$  is a compact subset of  $Y$ , and  $\varepsilon > 0$ .

Let  $G$  be the set of all autohomeomorphisms of  $X$  of the form  $g(x) = x \circ h$ , where  $h \in H$ , together with those of the form  $g(x) = \sum_{\gamma \in \Gamma} h_\gamma \circ x \circ I_\gamma$ , where for  $\gamma \in \Gamma$   $h_\gamma$  is an autoisometry of  $T$ .

By saturation and the definition of  $H$ , there is a  $*$ compact subset of  $K$  of  $*Y$  such that, for every standard compact  $K' \subseteq Y$  and every  $h \in H$ ,  $*K' \subseteq K = *h(K)$ . Put  $\varepsilon > 0$  infinitesimal, and let  $U = \{N(f, K, \varepsilon) : f \in *X\}$ . Since  $K$  is  ${}^\sigma H$ -invariant,  $U$  is  ${}^\sigma G$ -distributed; since  $K$  contains all standard compact sets and  $\varepsilon \approx 0$ , each  $u \in U$  is infinitesimal. It follows from Theorem 1 that there is a  $G$ -invariant Borel probability measure on  $X$ .

Could this measure have been obtained as a consequence of Corollary 1 or 2? Suppose  $H$  has the additional property (satisfied, for example, by finite permutations on a discrete  $Y$ ) that if  $K$  is compact then  $h(K) \cap K = \emptyset$  for some  $h \in H$ .

Fix  $t \in T$ , let  $\tau$  be the constant function  $\tau(y) = t$ , and let  $f$  be any element of  $X - \{\tau\}$  with the property that  $f(y) = t$  for all  $y$  off some compact subset  $K$  of  $Y$ .

Suppose  $u$  is any open neighborhood of  $\tau$ ; for some  $\varepsilon > 0$  and compact  $K' \subseteq Y$ ,  $N(\tau, K', \varepsilon) \subseteq u$ . Choose  $h \in H$  satisfying  $h(K \cup K') \cap (K \cup K') = \emptyset$ , and let  $g$  be the autohomeomorphism  $g(x) = x \circ h$ . Since  $g(\tau) = \tau$ ,  $g(N(\tau, K', \varepsilon)) = N(\tau, h(K'), \varepsilon)$ . Thus both  $\tau$  and  $f$  are elements of  $g(u)$ .

The disjoint pair of compact sets  $\{f\}$  and  $\{\tau\}$  provide an immediate counterexample to the hypotheses of Steinlage's theorem. As for Mycielski's theorem, observe that  $X$  is, in general, not metrizable. Moreover, in those cases where  $X$  is metrizable with a metric  $\eta$ , then some element of  $G$  takes the  $\delta$ -ball around  $\tau$ , where  $\delta = \eta(\tau, f)/2$ , onto an open set containing both  $\tau$  and  $f$ ; it follows that  $G$  is not a set of isometries, so the hypotheses of Mycielski's theorem fail as well.  $\square$

**5. Extensions.** In [6], Steinlage proves his theorem not just for compact Hausdorff spaces, but indeed for any *locally* compact Hausdorff space  $X$ . Theorem 1 extends to such spaces as well, provided the hypothesis is strengthened and the conclusion weakened as follows: In the hypothesis replace the word "compact" by the word "closed"; in the conclusion, replace "Borel" by "Baire". The proof is quite the same, though technically more involved. See [5] for a discussion of the special case where  $G$  is a group of autohomeomorphisms.

The author does not know whether, in the locally compact case, Theorem 1 holds with "compact" in place of "closed".

If  $X$  is a metric space and  $G$  is the set of local isometries, then Theorem 1 is no stronger than Mycielski's theorem. In particular, it gives no more information about the *Scottish book* problem mentioned in §1. That problem is to find a measure invariant under *all* partial isometries, not just those with open domain and range. Christoph Bandt has shown that such a measure exists provided the space  $X$  is locally homogeneous, i.e., for every  $x, y \in X$  there is a local isometry  $g$  of  $X$  with  $g(x) = y$ .

**6. Acknowledgments.** The author is grateful to Nigel Cutland for discussion and advice, and to the referee for pointing out a terminological ambiguity in the original manuscript and for describing Bandt's result mentioned above.

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