SHORT PROOFS OF THREE THEOREMS ON HARMONIC FUNCTIONS

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ABSTRACT. We present elementary proofs—shorter than any others that we know—for three related theorems.

THEOREM I. A function \( u \) that is harmonic and positive in the upper half-space \( \{x \in \mathbb{R}^n: x_n > 0\} \) and zero on the boundary hyperplane must be of the form \( ax_n \).

When \( n = 2 \), for instance, this implies that an entire function which maps the upper half-plane into itself and is real on the real axis is an affine function \( az + b \). Many proofs have been given, for instance [1–7]. Our proof is similar to the one given in [4] for the planar case.

THEOREM II. A function \( u \) that is harmonic in \( \mathbb{R}^n \) and bounded from one side by a polynomial must be a polynomial, and of no higher degree.

This is a strong form of Liouville's theorem.

THEOREM III. A function \( f \), meromorphic in the whole complex plane, real on the real axis, with \( \text{Im} \, f(z) \geq 0 \) when \( \text{Im} \, z \geq 0 \), has the form

\[
f(z) = b_0 + b_1 z + \frac{c}{z} + \sum \left( \frac{A_k}{z - a_k} + \frac{A_k}{a_k} \right)
\]

where \( b_0 \in \mathbb{R}, b_1 \geq 0, c \leq 0, a_k \in \mathbb{R} \setminus \{0\}, A_k < 0 \), and the series converges uniformly on every compact set that avoids the poles.

This is a theorem of Chebotarev and Meiman [1, p. 197].

PROOF OF THEOREM I. By the reflection principle we may assume that \( u \) is harmonic in the whole space \( \mathbb{R}^n \), and that \( u \) is an odd function of \( x_n \). Write

\[
u(x) = \sum_{j=1}^{\infty} u_j(x) = \sum_{j=1}^{\infty} r^j u_j(\omega)
\]

where \( x = r\omega, r = |x|, \) and each \( u_j \) is a homogeneous polynomial of degree \( j \). The \( u_j \) inherit from \( u \) the properties of harmonicity and anti-symmetry in the variable \( x_n \).

In particular \( u_1(x) = ax_n \), and the function \( |u_j(x)/x_n| \) extends continuously to the unit sphere and has some upper bound \( c_j \) there. Multiply (1) by \( g(\omega) = c_k \omega_n - u_k(\omega) \) and integrate over the unit sphere. Since spherical harmonics of...
different degrees are orthogonal, and $u$ and $g$ have the same sign at each point, we obtain
\[
0 \leq ac_k r \|\omega_n\|^2 - r^k \|u_k\|^2
\]
for each $k$. Since $r$ can be arbitrarily large, this implies that $u_k = 0$ when $k > 1$.

**Proof of Theorem II.** Changing signs if necessary we have
\[
0 < \frac{1}{2} k \omega_n - r^k \|u_k\|^2
\]
for each $k$. Since $r$ can be arbitrarily large, this implies that $u_k = 0$ when $k > 1$.

**Proof of Theorem II.** Changing signs if necessary we have
\[
0 < \frac{1}{2} k \omega_n - r^k \|u_k\|^2
\]
for each $k$. Since $u_k(\omega)$ is orthogonal to $p(r\omega)$ when $k > m$, the $u_k$ must vanish for such $k$. Hence $u$ is a polynomial of degree at most $m$.

Notice that this proof not only avoids using the Poisson formula, but also (when $n = 2$) establishes a strong form of Liouville's theorem without appealing to Cauchy's theorem: namely, if $f$ is entire and $\limsup_{z \to \infty} |f(z)|/|z|^m < \infty$ then $f$ is a polynomial of degree at most $m$. Indeed if $m$ is even this condition implies that the real and imaginary parts of $f$ are bounded by a polynomial of degree $m$ in $x$ and $y$, and so the conclusion follows from Theorem II; if $m$ is odd the same argument applies to the function $zf(z)$.

**Proof of Theorem III.** Since $f$ is real on the real axis, $f(\bar{z}) = \overline{f(z)}$ and hence $\text{Im}\ f(z) \leq 0$ when $\text{Im}\ z \leq 0$. Since $f$ maps a neighborhood of a pole to a neighborhood of $\infty$, no neighborhood of a pole can lie entirely in either the upper or the lower half-plane: the poles must lie on the real axis. Near a pole $p$ the function $f$ behaves like $C(z - p)^{-m}$, and since the upper half of a neighborhood of $p$ maps into the upper half-plane, $m = 1$ and $C < 0$.

By composing $f$ with a real translation we may assume that $0$ is not a pole. Let $a_1, a_2, \ldots$ be the poles of $f$ in increasing order of absolute value, and let $A_k$ be the (negative) residue of $f$ at $a_k$. In the annulus $\{z: |a_n| < |z| < |a_{n+1}|\}$ the function $f$ has a Laurent series $f(z) = \sum_{\infty} \beta_j z^j$ with real coefficients $\beta_j$, and in particular
\[
\beta_{-1} = \sum_{k \leq n} A_k, \quad \beta_1 = f'(0) + \sum_{k \leq n} A_k/a_k^2.
\]
If $|a_n| < r < |a_{n+1}|$ then by hypothesis
\[
0 \leq \text{Im} \int_0^{2\pi} f(re^{i\theta}) \sin \theta \, d\theta = \pi \beta_1 r - \pi \beta_{-1} r^{-1},
\]
and so
\[
0 \leq \sum_{k \leq n} |A_k| \left( a_k^{-2} - r^{-2} \right) \leq f'(0).
\]
Letting $r \to \infty$ we conclude that $\sum |A_k|/a_k^2 \leq f'(0)$.

The convergence of $\sum A_k/a_k^2$ implies that
\[
f(z) = g(z) + \sum \left( \frac{A_k}{z-a_k} + \frac{A_k}{a_k} \right)
\]
where the sum converges uniformly on compact subsets of \( C \setminus \{a_k\} \) and \( g(z) = \sum_{j=0}^{\infty} b_j z^j \) is entire (and \( b_j \in \mathbb{R} \)). Let \( h_n(\theta) = n \sin \theta \pm \sin n\theta \). By expanding in binomial series we find for \( r \neq |a_k| \) that

\[
\text{Im} \int_0^{2\pi} \frac{A_k}{re^{i\theta} - a_k} h_n(\theta) \, d\theta \leq \pi (n + 1)r|A_k|/a_k^2.
\]

Since \( h_n(\theta) \) has the same sign as \( \sin \theta \), we have by hypothesis

\[
0 \leq \pi^{-1} \text{Im} \int_0^{2\pi} f(re^{i\theta})h_n(\theta) \, d\theta \leq nb_1 r \pm b_n r^n + (n + 1)r \sum |A_k|/a_k^2.
\]

Therefore \( b_n = 0 \) for \( n \geq 2 \). Setting \( z = 0 \) shows that \( b_0 \) is real. Moreover \( f''(0) = b_1 + \sum |A_k|/a_k^2 \leq b_1 + f'(0) \), so \( b_1 \geq 0 \). This completes the proof.

**REFERENCES**


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