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SHORT PROOFS OF THREE THEOREMS ON HARMONIC FUNCTIONS

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ABSTRACT. We present elementary proofs—shorter than any others that we know—for three related theorems.

THEOREM I. *A function u that is harmonic and positive in the upper half-space $\{x \in \mathbf{R}^n\colon\ x_n > 0\}$ and zero on the boundary hyperplane must be of the form ax_n.*

When $n = 2$, for instance, this implies that an entire function which maps the upper half-plane into itself and is real on the real axis is an affine function $az + b$. Many proofs have been given, for instance [**1**–**7**]. Our proof is similar to the one given in [**4**] for the planar case.

THEOREM II. *A function u that is harmonic in $\mathbf{R}^n$ and bounded from one side by a polynomial must be a polynomial, and of no higher degree.*

This is a strong form of Liouville's theorem.

THEOREM III. *A function f, meromorphic in the whole complex plane, real on the real axis, with $\operatorname{Im} f(z) \geq 0$ when $\operatorname{Im} z \geq 0$, has the form*

$$f(z) = b_0 + b_1 z + \frac{c}{z} + \sum \left(\frac{A_k}{z - a_k} + \frac{A_k}{a_k} \right)$$

where $b_0 \in \mathbf{R}$, $b_1 \geq 0$, $c \leq 0$, $a_k \in \mathbf{R} \setminus \{0\}$, $A_k < 0$, and the series converges uniformly on every compact set that avoids the poles.

This is a theorem of Chebotarev and Meĭman [**1**, p. 197].

PROOF OF THEOREM I. By the reflection principle we may assume that u is harmonic in the whole space $\mathbf{R}^n$, and that u is an odd function of x_n. Write

$$u(x) = \sum_{j=1}^{\infty} u_j(x) = \sum_{j=1}^{\infty} r^j u_j(\omega) \tag{1}$$

where $x = r\omega$, $r = |x|$, and each u_j is a homogeneous polynomial of degree j. The u_j inherit from u the properties of harmonicity and anti-symmetry in the variable x_n. In particular $u_1(x) = ax_n$, and the function $|u_j(x)/x_n|$ extends continuously to the unit sphere and has some upper bound c_j there. Multiply (1) by $g(\omega) = c_k\omega_n - u_k(\omega)$ and integrate over the unit sphere. Since spherical harmonics of

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different degrees are orthogonal, and u and g have the same sign at each point, we obtain

$$0 \leq ac_k r\|\omega_n\|^2 - r^k\|u_k\|^2$$

for each k. Since r can be arbitrarily large, this implies that $u_k = 0$ when $k > 1$.

PROOF OF THEOREM II. Changing signs if necessary we have

$$0 \leq p(r\omega) + \sum_{j=0}^{\infty} r^j u_j(\omega) \tag{2}$$

where p is a polynomial, say of degree m, and each u_j is a homogeneous harmonic polynomial of degree j. Let b_j be an upper bound for $|u_j(x)|$ on the unit sphere. Multiplying (2) by $b_k - u_k(\omega)$ and integrating over the unit sphere gives

$$0 \leq O(r^m) - \int p(r\omega)u_k(\omega) - r^k\|u_k\|^2$$

for each k. Since $u_k(\omega)$ is orthogonal to $p(r\omega)$ when $k > m$, the u_k must vanish for such k. Hence u is a polynomial of degree at most m.

Notice that this proof not only avoids using the Poisson formula, but also (when $n = 2$) establishes a strong form of Liouville's theorem without appealing to Cauchy's theorem: namely, if f is entire and $\limsup_{z\to\infty} |f(z)|/|z|^m < \infty$ then f is a polynomial of degree at most m. Indeed if m is even this condition implies that the real and imaginary parts of f are bounded by a polynomial of degree m in x and y, and so the conclusion follows from Theorem II; if m is odd the same argument applies to the function $zf(z)$.

PROOF OF THEOREM III. Since f is real on the real axis, $f(\bar{z}) = \overline{f(z)}$ and hence $\operatorname{Im} f(z) \leq 0$ when $\operatorname{Im} z \leq 0$. Since f maps a neighborhood of a pole to a neighborhood of ∞, no neighborhood of a pole can lie entirely in either the upper or the lower half-plane: the poles must lie on the real axis. Near a pole p the function f behaves like $C(z-p)^{-m}$, and since the upper half of a neighborhood of p maps into the upper half-plane, $m = 1$ and $C < 0$.

By composing f with a real translation we may assume that 0 is not a pole. Let $a_1, a_2, \ldots$ be the poles of f in increasing order of absolute value, and let A_k be the (negative) residue of f at a_k. In the annulus $\{z\colon |a_n| < |z| < |a_{n+1}|\}$ the function f has a Laurent series $f(z) = \sum_{-\infty}^{\infty} \beta_j z^j$ with real coefficients β_j, and in particular

$$\beta_{-1} = \sum_{k\leq n} A_k, \quad \beta_1 = f'(0) + \sum_{k\leq n} A_k/a_k^2.$$

If $|a_n| < r < |a_{n+1}|$ then by hypothesis

$$0 \leq \operatorname{Im} \int_0^{2\pi} f(re^{i\theta}) \sin\theta \, d\theta = \pi\beta_1 r - \pi\beta_{-1} r^{-1},$$

and so

$$\sum_{k\leq n} |A_k| \left(a_k^{-2} - r^{-2}\right) \leq f'(0).$$

Letting $r \to \infty$ we conclude that $\sum |A_k|/a_k^2 \leq f'(0)$.

The convergence of $\sum A_k/a_k^2$ implies that

$$f(z) = g(z) + \sum \left(\frac{A_k}{z - a_k} + \frac{A_k}{a_k}\right)$$

where the sum converges uniformly on compact subsets of $\mathbf{C} \setminus \{a_k\}$ and $g(z) = \sum_0^\infty b_j z^j$ is entire (and $b_j \in \mathbf{R}$). Let $h_n(\theta) = n \sin\theta \pm \sin n\theta$. By expanding in binomial series we find for $r \neq |a_k|$ that

$$\operatorname{Im} \int_0^{2\pi} \frac{A_k}{re^{i\theta} - a_k} h_n(\theta)\, d\theta \leq \pi(n+1)r|A_k|/a_k^2.$$

Since $h_n(\theta)$ has the same sign as $\sin\theta$, we have by hypothesis

$$0 \leq \pi^{-1} \operatorname{Im} \int_0^{2\pi} f(re^{i\theta}) h_n(\theta)\, d\theta \leq nb_1 r \pm b_n r^n + (n+1)r \sum |A_k|/a_k^2.$$

Therefore $b_n = 0$ for $n \geq 2$. Setting $z = 0$ shows that b_0 is real. Moreover $f'(0) = b_1 + \sum |A_k|/a_k^2 \leq b_1 + f'(0)$, so $b_1 \geq 0$. This completes the proof.

References

1. N. G. Chebotarev and N. N. Meĭman, *The Routh-Hurwitz problem for polynomials and entire functions*, Trudy Mat. Inst. Steklov. **26** (1949). (Russian)
2. A. Dinghas, *Über positive harmonische Funktionen in einem Halbraum*, Math. Z. **46** (1940), 559–570.
3. Ü. Kuran, *Harmonic majorizations in half-balls and half-spaces*, Proc. London Math. Soc. **21** (1970), 614–636.
4. L. H. Loomis and D. V. Widder, *The Poisson integral representation of functions which are positive and harmonic in a half-plane*, Duke Math. J. **9** (1942), 643–645.
5. M. Tideman, *Elementary proof of a uniqueness theorem for positive harmonic functions*, Nordisk. Mat. Tidskrift **2** (1954), 95–96.
6. N. Tschebotareff, *Über Realität von Nullstellen ganzer transzendenter Funktionen*, Math. Ann. **99** (1928), 660–686.
7. M. Tsuji, *On a positive harmonic function in a half-plane*, Japan. J. Math. **15** (1939), 277–285.

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