HYPONORMAL POWERS OF COMPOSITION OPERATORS
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ABSTRACT. Let $T_i$, $i = 1, 2$, be measurable transformations which define bounded composition operators $C_{T_i}$ on $L^2$ of a $\sigma$-finite measure space. Denote their respective Radon-Nikodym derivatives by $h_i$, $i = 1, 2$. The main result of this paper is that if $h_j \circ T_i \leq h_j$, $i, j = 1, 2$, then for each of the positive integers $m, n, p$ the operator $[C_{T_1}^m C_{T_2}^n]^p$ is hyponormal. As a consequence, we see that the sufficient condition established by Harrington and Whitley for hyponormality of a composition operator is actually sufficient for all powers to be hyponormal.

I. Preliminaries. Let $(X, \Sigma, m)$ be a $\sigma$-finite measure space, and let $T$ be a measurable transformation from $X$ into itself. Set $L^2 = L^2(X, \Sigma, m)$. The equation

\[(0) \quad C_T f = f \circ T, \quad f \in L^2 \]

defines a composition transformation from $L^2$ to the space of $\mathbb{C}$-valued functions on $X$. $C_T$ is a bounded linear operator on $L^2$ precisely when (i) $m \circ T^{-1}$ is absolutely continuous with respect to $m$, and (ii) $h = d(m \circ T^{-1})/dm$ is in $L^\infty(X, \Sigma, m) = L^\infty$. Denote by $R(C_T)$ the range of $C_T$, by $C_T^*$ the adjoint of $C_T$, and define $T^{-1}(\Sigma)$ as the relative completion of the $\sigma$-algebra $\{T^{-1}(A) : A \in \Sigma\}$. The following lemma, due to Harrington and Whitley [2, p. 126], is well known and useful.

1.1 LEMMA. Let $P$ denote the projection of $L^2$ onto $R(C_T)$.
(a) $C_T^* C_T f = h f$ and $C_T C_T^* f = (h \circ T) P f$, all $f \in L^2$.
(b) $\overline{R(C_T)} = \{f \in L^2 : f \text{ is } T^{-1}(\Sigma)-\text{measurable}\}$.
(c) If $f$ is $T^{-1}(\Sigma)$-measurable, and $g$ and $fg$ belong to $L^2$, then
\[(3) \quad P(fg) = fP(g). \]
(f need not be in $L^2$.)

Finally we recall that a bounded linear operator $A$ on Hilbert space $\mathcal{H}$ is hyponormal if $A^* A - AA^* \geq 0$ (or equivalently, if $\|Af\| \geq \|A^* f\|$, for each $f \in \mathcal{H}$.) Now let $T_1$ and $T_2$ be measurable transformations of $X$ with R-N derivatives $h_1$ and $h_2$, respectively, with (i) and (ii) above satisfied by both pairs. It follows that the transformation $T_3 : X \to X$ given by $T_1 \circ T_2$ will be measurable, and the R-N
derivative $h_3$ is given by $h_3 = (h_1)(g)$, where $g$ is the unique $\Sigma$-measurable function satisfying $g \circ T_i = E(h_2 | T_i^{-1}(\Sigma))$. (Here, $E(\cdot | T_i^{-1}(\Sigma))$ denotes the conditional expectation of $\cdot$ with respect to the sub-$\sigma$-algebra $T_i^{-1}(\Sigma)$.) $T_3$ and $h_3$ will satisfy conditions (i) and (ii) also. It is shown in [2] that $h \circ T \leq h$ a.e. is sufficient for the hyponormality of $C_T$. The main result of this paper is that if $h_i \circ T_i \leq h_j$ for $i, j = 1, 2$, then for each of the positive integers $m, n, p$ the operator $[C_T^m C_T^n]^p$ is hyponormal. As a consequence, we see that Harrington and Whitley’s sufficient condition for hyponormality of a composition operator is actually sufficient for all powers to be hyponormal. In the last section of this paper, we give an example of a hyponormal composition operator whose square is not hyponormal.

We close this section with a useful observation. In [2, p. 130], Harrington and Whitley show that for every $g \in L^2$

$$\langle (h \circ T) P g, g \rangle \geq \langle (h \circ T) g, g \rangle,$$

where $T$ and $h$ are as above. Similar calculations may be used to prove the following lemma, whose proof we omit.

1.2 LEMMA. With $h, T$, and $P$ as above,

$$\langle (h^n \circ T) P g, g \rangle \leq \langle (h^n \circ T) g, g \rangle, \quad n \in \mathbb{N}, g \in L^2.$$

II. Main result and corollaries. In order to prove our main result (Theorem 1), it is necessary to state and prove several lemmas. They are stated so that each lemma depends on some subset of the previously stated ones. The proofs which are given contain the essential ideas, and may easily be adjusted to give the ones we omit.

2.1 LEMMA. If $h \circ T \leq h$ a.e. $m$, then for all $n \in \mathbb{N}$, $f \in L^2$ we have

$$\langle h^n f, f \rangle \leq \langle (C_T^n)^* C_T^n f, f \rangle.$$

PROOF. For $n = 1$ the lemma is true by (1). Suppose (6) holds for $n = 1, 2, \ldots, k$ and all $f \in L^2$. Then

$$\langle (C_T^{k+1})^* C_T^{k+1} f, f \rangle = \langle (C_T^k C_T) C_T C_T f, f \rangle = \langle (C_T^n)^* C_T^n (C_T f), (C_T f) \rangle \geq \langle h^n C_T f, C_T f \rangle,$$

by the inductive hypothesis. But $\langle h^n C_T f, C_T f \rangle = \int h^n (|f|^2 \circ T) dm$. By hypothesis, $h \geq h \circ T$ (a.e.), so that $h^n \geq h^n \circ T$ a.e., and we have

$$\langle h^n C_T f, C_T f \rangle \geq \int (h^n \circ T) (|f|^2 \circ T) dm = \int (h^n |f|^2) h dm = \langle h^{n+1} f, f \rangle,$$

so that (6) holds for $n = k + 1$, and (2.1) follows by induction. □

2.2 LEMMA. If $h \circ T \leq h$ a.e., then for all $n \in \mathbb{N}$, $f \in L^2$, we have

$$\langle C_T^n (C_T^n)^* f, f \rangle \leq \langle (h \circ T)^n f, f \rangle \leq \langle h^n f, f \rangle.$$

PROOF. Again we will induct on $n$. For $n = 1$ and $f \in L^2$,

$$\langle C_T C_T f, f \rangle = \langle (h \circ T) P f, f \rangle \leq \langle (h \circ T) f, f \rangle,$$

by Lemma 1.2. But

$$\langle (h \circ T) f, f \rangle = \int (h \circ T) |f|^2 dm \leq \int h |f|^2 dm = \langle h f, f \rangle,$$
since $0 \leq h \circ T \leq h$ a.e. Calculations similar to those used in the proof of Lemma 2.1 complete this proof by induction. □

Now with $T_1, T_2$ as above, we set $A = C_{T_1}$ and $B = C_{T_2}$, so that the product $AB$ is the operator $C_{T_3}$.

2.3 LEMMA. With $A, B, h_1,$ and $h_2$ as above, if

(10) \[ \begin{align*} (a) & \quad h_2 \circ T_2 \leq h_1 \text{ a.e.,} & (b) & \quad h_1 \circ T_1 \leq h_1 \text{ a.e.,} \end{align*} \]

then for each $m, n \in \mathbb{N}$ and $f \in L^2$,

(11) \[ \langle (A^m B^n)^*(A^m B^n)f, f \rangle \geq \langle h_2^{m+n}f, f \rangle. \]

PROOF. First we prove

2.3.1 Claim. With $T$ and $h$ as in §1, for all $r, m \in \mathbb{N}$ and $f \in L^2$, we have

(12) \[ \langle (h \circ T)^r(C_T^m f, C_T^m f) \rangle = \langle h^{r+m}f, f \rangle. \]

Proof of claim. Fix $r$ and induct on $m$. For $m = 1$ and $f \in L^2$,

\[ \begin{align*} \langle (h \circ T)^r C_T f, C_T f \rangle &= \int (h \circ T)^r (|f|^2 \circ T) \, dm \\ &= \int (h^r |f|^2) \, dm = \langle h^{r+1} f, f \rangle. \end{align*} \]

Suppose (12) holds for $m = k$ and all $f \in L^2$. Then

\[ \begin{align*} \langle (h \circ T)^r C_T^{k+1} f, C_T^{k+1} f \rangle &= \langle (h \circ T)^{r+k} C_T f, C_T f \rangle \\ &= \int (h^{r+k} \circ T)(|f|^2 \circ T) \, dm = \langle (h^{r+k+1} f, f \rangle, \end{align*} \]

and (12) holds for $m = k + 1$. The claim is proved by induction.

To finish the proof of Lemma 2.3, observe that

\[ \begin{align*} \langle (A^m B^n)^*(A^m B^n)f, f \rangle &= \langle ((A^m)^*(A^m)) B^n f, B^n f \rangle \\ &\geq \langle h_1^m B^n f, f \rangle \quad \text{(by Lemma 2.1)} \\ &\geq \langle h_2^m B^n f, B^n f \rangle \quad \text{(by hypothesis)} \\ &= \langle h_2^{m+n} f, f \rangle \quad \text{(by Claim 2.3.1).} \quad \Box \]

2.4 LEMMA. With $A, B, h_1,$ and $h_2$ as above, if

(13) \[ \begin{align*} h_i \circ T_i &\leq h_j, \quad i, j = 1, 2, \end{align*} \]

then for each $m, n \in \mathbb{N}$ and $f \in L^2$,

(14) \[ \langle (A^m B^n)(A^m B^n)^* f, f \rangle \leq \langle h_2^{m+n} f, f \rangle. \]

PROOF. First we prove

2.4.1 Claim. If $h \circ T \leq h$ a.e., then for all $r, m \in \mathbb{N}$, $f \in L^2$,

(15) \[ \langle h^r(C_T^m)^* f, (C_T^m)^* f \rangle \leq \langle (h \circ T)^{r+m} f, f \rangle. \]

Proof of claim. Fix $r$ and induct on $m$. For $m = 1$ and $f \in L^2$,

\[ \begin{align*} \langle h^r C_T^* f, C_T^* f \rangle &= \langle (h^r \circ T) C_T^* f, C_T^* f \rangle \\ &= \langle (h^{r+1} \circ T) P f, f \rangle \quad \text{(by Lemma 1.1)} \\ &\leq \langle (h^{r+1} \circ T) f, f \rangle \quad \text{(by Lemma 1.2).} \end{align*} \]
Completing the induction step is similar and is left to the reader. To finish the proof of Lemma 2.4, note that

\[
( (A^m B^n)(A^m B^n)^* f, f ) = ( (B^n(A^m)^*) (A^m)^* f, (A^m)^* f ) \leq (h_2 \circ T_2)^n (A^m)^* f, (A^m)^* f ) \text{ (by Lemma 2.2) } \\
\leq (h_1^n (A^m)^* f, (A^m)^* f ) \text{ (by hypothesis) } \\
\leq ( (h_1 \circ T_1)^n f, f ) \text{ (by Claim 2.4.1) } \\
\leq (h_2^{n+m} f, f ) \text{ (by hypothesis). } \qed
\]

Finally, similar techniques and the above lemmas may be used to prove the following pair of inequalities, which we collect as Lemma 2.5 and state without proof.

2.5 LEMMA. If (13) holds, then for all \( m, n, p \in \mathbb{N} \) and \( f \in L^2 \), we have

\[
(16) \quad ( [A^m B^n]^p (A^m B^n)^p f, f ) \geq (h_2^{m+n} p f, f ) \\
\]

and

\[
(17) \quad ( [A^m B^n]^p (A^m B^n)^p )^* f, f ) \leq (h_2^{m+n} p f, f ). \quad \square
\]

REMARK. \( h_2 \circ T_2 \leq h_2 \) is not necessary in the proof of (16). Now we may easily prove our main result:

THEOREM 1. If (13) holds, \( (A^m B^n)^p \) is hyponormal for all \( m, n, p \in \mathbb{N} \).

PROOF. By first applying (16) and then applying (17) we have, for \( m, n, p \in \mathbb{N} \) and \( f \in L^2 \),

\[
( [A^m B^n]^p (A^m B^n)^p f, f ) \geq (h_2^{m+n} p f, f ) \geq ( [A^m B^n]^p [ (A^m B^n)^p ] f, f ),
\]

so that \( ( [A^m B^n]^p f ) \geq ( [A^m B^n]^p f ) \). \( \square \)

The following corollaries are immediate.

COROLLARY 1. With \( A, B, h_1, \) and \( h_2 \) as above, if (13) holds, then for each \( m, n \in \mathbb{N} \), \( A^m B^n \) is hyponormal. \( \square \)

REMARK. This may be proved directly using Lemmas 2.3 and 2.4.

COROLLARY 2. If (13) holds, then \( (AB)^p \) is hyponormal for each \( p \in \mathbb{N} \).

REMARK. Observe that (13) implies a priori that both \( A \) and \( B \) are hyponormal. Actually, the conclusion of Corollary 2 is true under the weaker hypothesis that \( h_i \circ T_i \leq h_j, i \neq j, i, j = 1, 2 \). These hypotheses may be used to prove the inequalities

\[
(18) \quad ( [AB]^p [AB]^p f, f ) \geq (h_2^{2p} f, f ) \\
\]

and

\[
(19) \quad (AB)^p [AB]^p f, f ) \leq (h_2^{2p} f, f ),
\]

from which Corollary 2 follows immediately.

COROLLARY 3. With \( T, h, \) and \( m \) as in \( \S 1 \), if \( h \circ T \leq h \) a.e. \( m \), then \( C_T^n \) is hyponormal for each \( n \in \mathbb{N} \). \( \square \)

REMARK. Corollary 3 also follows directly from Lemmas 2.1 and 2.2.
III. An example. Halmos notes in [1] that it is not easy to find examples of hyponormal operators whose squares are not hyponormal. The simplest such example is due to Ito and Wong [3]; namely, that $U^* + 2U$, where $U$ is the unilateral shift on $l^2(N)$, is hyponormal but its square is not. For a hyponormal composition operator whose square is not hyponormal, we have the following Example. Let $X = [0, \infty)$ be equipped with Lebesgue measure on the Borel sets. Set

$$Tx = x\chi_{[0,1)} + (10x/3 - 10/3)\chi_{(1,13/10]} + (5x/2 - 9/4)\chi_{(13/10,3/2]}$$

$$+ (3x/10 - 9/20)\chi_{(3/2,29/6]} + (x/100 + 871/600)\chi_{(29/6,\infty)}.$$ 

Then

$$h = dm o T^{-1}/dm = 139/30\chi_{[0,1)} + 2/5\chi_{(1,3/2]} + 100\chi_{(3/2,\infty)}.$$ 

($\chi_{[a,b)}$ stands for the characteristic function of $[a,b)$.) Using Lemma 15 of [2], it can be shown that $C_T$ is hyponormal. If one considers $g(x) = x\chi_{(1,13/10]}$, then a direct but tedious computation shows that $(C_T^{2*}C_T^2g,g) < (C_T^2C_T^{2*}g,g)$ so that $C_T^2$ is not hyponormal.

BIBLIOGRAPHY


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