PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 102, Number 4, April 1988

HYPONORMAL POWERS OF COMPOSITION OPERATORS

PHILLIP DIBRELL AND JAMES T. CAMPBELL

(Communicated by John B. Conway)

ABSTRACT. Let T_i, $i = 1, 2$, be measurable transformations which define bounded composition operators C_{T_i} on L^2 of a σ-finite measure space. Denote their respective Radon-Nikodym derivatives by h_i, $i = 1, 2$. The main result of this paper is that if $h_i \circ T_i \leq h_j$, $i, j = 1, 2$, then for each of the positive integers m, n, p the operator $[C_{T_1}^m C_{T_2}^n]^p$ is hyponormal. As a consequence, we see that the sufficient condition established by Harrington and Whitley for hyponormality of a composition operator is actually sufficient for all powers to be hyponormal.

I. Preliminaries. Let (X, Σ, m) be a σ-finite measure space, and let T be a measurable transformation from X into itself. Set $L^2 = L^2(X, \Sigma, m)$. The equation

$$C_T f = f \circ T, \qquad f \in L^2 \tag{0}$$

defines a composition transformation from L^2 to the space of $\mathbf{C}$-valued functions on X. C_T is a bounded linear operator on L^2 precisely when (i) $m \circ T^{-1}$ is absolutely continuous with respect to m, and (ii) $h = dm \circ T^{-1}/dm$ is in $L^\infty(X, \Sigma, m) = L^\infty$. Denote by $R(C_T)$ the range of C_T, by C_T^* the adjoint of C_T, and define $T^{-1}(\Sigma)$ as the relative completion of the σ-algebra $\{T^{-1}(A)\colon A \in \Sigma\}$. The following lemma, due to Harrington and Whitley [**2**, p. 126], is well known and useful.

1.1 LEMMA. *Let P denote the projection of L^2 onto $\overline{R(C_T)}$.*

(a)

$$C_T^* C_T f = hf \quad \textit{and} \quad C_T C_T^* f = (h \circ T) P f, \quad \textit{all } f \in L^2. \tag{1}$$

(b)

$$\overline{R(C_T)} = \{f \in L^2 \colon f \textit{ is } T^{-1}(\Sigma)\textit{-measurable}\}. \tag{2}$$

(c) *If f is $T^{-1}(\Sigma)$-measurable, and g and fg belong to L^2, then*

$$P(fg) = f P(g). \tag{3}$$

(*f need not be in L^2.*) □

Finally we recall that a bounded linear operator A on Hilbert space $\mathcal{H}$ is *hyponormal* if $A^*A - AA^* \geq 0$ (or equivalently, if $\|Af\| \geq \|A^*f\|$, for each $f \in \mathcal{H}$.) Now let T_1 and T_2 be measurable transformations of X with R-N derivatives h_1 and h_2, respectively, with (i) and (ii) above satisfied by both pairs. It follows that the transformation $T_3\colon X \to X$ given by $T_1 \circ T_2$ will be measurable, and the R-N

Received by the editors July 1, 1986 and, in revised form, September 22, 1986 and January 26, 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 47B20; Secondary 47B38.

derivative h_3 is given by $h_3 = (h_1)(g)$, where g is the unique Σ-measurable function satisfying $g \circ T_1 = E(h_2 \mid T_1^{-1}(\Sigma))$. (Here, $E(\cdot|T_1^{-1}(\Sigma))$ denotes the conditional expectation of $\cdot$ with respect to the sub-σ-algebra $T_1^{-1}(\Sigma)$.) T_3 and h_3 will satisfy conditions (i) and (ii) also. It is shown in [**2**] that $h \circ T \leq h$ a.e. is sufficient for the hyponormality of C_T. The main result of this paper is that if $h_1 \circ T_1 \leq h_j$ for $i, j = 1, 2,$, then for each of the positive integers m, n, and p the operator $[C_{T_1}^m C_{T_2}^n]^p$ is hyponormal. As a consequence, we see that Harrington and Whitley's sufficient condition for hyponormality of a composition operator is actually sufficient for all powers to be hyponormal. In the last section of this paper, we give an example of a hyponormal composition operator whose square is not hyponormal.

We close this section with a useful observation. In [**2**, p. 130], Harrington and Whitley show that for every $g \in L^2$

$$\langle (h \circ T)Pg, g\rangle \geq \langle (h \circ T)g, g\rangle, \tag{4}$$

where T and h are as above. Similar calculations may be used to prove the following lemma, whose proof we omit.

1.2 LEMMA. *With h, T, and P as above,*

$$\langle (h^n \circ T)Pg, g\rangle \leq (h^n \circ T)g, g\rangle, \qquad n \in \mathbf{N}, g \in L^2. \tag{5}$$

II. Main result and corollaries. In order to prove our main result (Theorem 1), it is necessary to state and prove several lemmas. They are stated so that each lemma depends on some subset of the previously stated ones. The proofs which are given contain the essential ideas, and may easily be adjusted to give the ones we omit.

2.1 LEMMA. *If $h \circ T \leq h$ a.e. m, then for all $n \in \mathbf{N}$, $f \in L^2$ we have*

$$\langle h^n f, f\rangle \leq \langle (C_T^n)^* C_T^n f, f\rangle. \tag{6}$$

PROOF. For $n = 1$ the lemma is true by (1). Suppose (6) holds for $n = 1, 2, \ldots, k$ and all $f \in L^2$. Then

$$\begin{aligned}\langle (C_T^{k+1})^* C_T^{k+1} f, f\rangle &= \langle (C_T^k C_T)^* C_T^k C_T f, f\rangle = \langle (C_T^k)^* C_T^k (C_T f), (C_T f)\rangle \\ &\geq \langle h^k C_T f, C_T f\rangle, \quad \text{for each } f \in L^2,\end{aligned}$$

by the inductive hypothesis. But $\langle h^k C_T f, C_T f\rangle = \int h^k(|f|^2 \circ T)\, dm$. By hypothesis, $h \geq h \circ T$ (≥ 0) a.e., so that $h^k \geq h^k \circ T$ a.e., and we have

$$\langle h^k C_T f, C_T f\rangle \geq \int (h^k \circ T)(|f|^2 \circ T)\, dm = \int (h^k |f|^2) h\, dm = \langle h^{k+1} f, f\rangle,$$

so that (6) holds for $n = k+1$, and (2.1) follows by induction. □

2.2 LEMMA. *If $h \circ T \leq h$ a.e., then for all $n \in \mathbf{N}$, $f \in L^2$, we have*

$$\langle C_T^n (C_T^n)^* f, f\rangle \leq \langle (h \circ T)^n f, f\rangle \leq \langle h^n f, f\rangle. \tag{7}$$

PROOF. Again we will induct on n. For $n = 1$ and $f \in L^2$,

$$\langle C_T C_T^* f, f\rangle = \langle (h \circ T)Pf, f\rangle \leq \langle (h \circ T)f, f\rangle,$$

by Lemma 1.2. But

$$\langle (h \circ T)f, f\rangle = \int (h \circ T)|f|^2\, dm \leq \int h|f|^2\, dm = \langle hf, f\rangle,$$

since $0 \leq h \circ T \leq h$ a.e. Calculations similar to those used in the proof of Lemma 2.1 complete this proof by induction. □

Now with T_1, T_2 as above, we set $A = C_{T_1}$ and $B = C_{T_2}$, so that the product AB is the operator C_{T_3}.

2.3 LEMMA. *With A, B, h_1, and h_2 as above, if*

$$\text{(10)} \qquad \text{(a)} \quad h_2 \circ T_2 \leq h_1 \text{ a.e.,} \quad \text{and}$$
$$\text{(b)} \quad h_1 \circ T_1 \leq h_1 \text{ a.e.,}$$

then for each m, $n \in \mathbf{N}$ and $f \in L^2$,

$$\text{(11)} \qquad \langle (A^m B^n)^*(A^m B^n) f, f\rangle \geq \langle h_2^{m+n} f, f\rangle.$$

PROOF. First we prove

2.3.1 *Claim.* With T and h as in §I, for all r, $m \in \mathbf{N}$ and $f \in L^2$, we have

$$\text{(12)} \qquad \langle (h \circ T)^r C_T^m f, C_T^m f\rangle = \langle h^{r+m} f, f\rangle.$$

Proof of claim. Fix r and induct on m. For $m = 1$ and $f \in L^2$,

$$\begin{aligned} \langle (h \circ T)^r C_T f, C_T f\rangle &= \int (h \circ T)^r (|f|^2 \circ T)\, dm \\ &= \int (h^r |f|^2) h\, dm = \langle h^{r+1} f, f\rangle. \end{aligned}$$

Suppose (12) holds for $m = k$ and all $f \in L^2$. Then

$$\begin{aligned} \langle (h \circ T)^r C_T^{k+1} f, C_T^{k+1} f\rangle &= \langle (h \circ T)^{r+k} C_T f, C_T f\rangle \\ &= \int (h^{r+k} \circ T)(|f|^2 \circ T)\, dm = \langle (h)^{r+k+1} f, f\rangle, \end{aligned}$$

and (12) holds for $m = k + 1$. The claim is proved by induction.

To finish the proof of Lemma 2.3, observe that

$$\begin{aligned} \langle (A^m B^n)^*(A^m B^n) f, f\rangle &= \langle [(A^m)^*(A^m)] B^n f, B^n f\rangle \\ &\geq \langle h_1^m B^n f, f\rangle \quad \text{(by Lemma 2.1)} \\ &\geq \langle (h_2 \circ T_2)^m B^n f, B^n f\rangle \quad \text{(by hypothesis)} \\ &= \langle h_2^{m+n} f, f\rangle \quad \text{(by Claim 2.3.1).} \quad \square \end{aligned}$$

2.4 LEMMA. *With A, B, h_1, and h_2 as above, if*

$$\text{(13)} \qquad h_i \circ T_i \leq h_j, \qquad i, j = 1, 2,$$

then for each m, $n \in \mathbf{N}$ and $f \in L^2$,

$$\text{(14)} \qquad \langle (A^m B^n)(A^m B^n)^* f, f\rangle \leq \langle h_2^{m+n} f, f\rangle.$$

PROOF. First we prove

2.4.1 *Claim.* If $h \circ T \leq h$ a.e., then for all $r, m \in \mathbf{N}$, $f \in L^2$,

$$\text{(15)} \qquad \langle h^r (C_T^m)^* f, (C_T^m)^* f\rangle \leq \langle (h \circ T)^{r+m} f, f\rangle.$$

Proof of claim. Fix r and induct on m. For $m = 1$ and $f \in L^2$,

$$\begin{aligned} \langle h^r C_T^* f, C_T^*, f\rangle &= \langle (h^r \circ T) C_T C_T^* f, f\rangle \\ &= \langle (h^{r+1} \circ T) P f, f\rangle \quad \text{(by Lemma 1.1)} \\ &\leq (h^{r+1} \circ T) f, f\rangle \quad \text{(by Lemma 1.2).} \end{aligned}$$

Completing the induction step is similar and is left to the reader. To finish the proof of Lemma 2.4, note that

$$\begin{aligned}\langle (A^m B^n)(A^m B^n)^* f, f\rangle &= \langle [B^n (B^n)^*](A^m)^* f, (A^m)^* f\rangle \\ &\leq \langle (h_2 \circ T_2)^n (A^m)^* f, (A^m)^* f\rangle \quad \text{(by Lemma 2.2)} \\ &\leq \langle h_1^n (A^m)^* f, (A^m)^* f\rangle \quad \text{(by hypothesis)} \\ &\leq \langle (h_1 \circ T_1)^{n+m} f, f\rangle \quad \text{(by Claim 2.4.1)} \\ &\leq \langle h_2^{n+m} f, f\rangle \quad \text{(by hypothesis).} \quad \square\end{aligned}$$

Finally, similar techniques and the above lemmas may be used to prove the following pair of inequalities, which we collect as Lemma 2.5 and state without proof.

2.5 LEMMA. *If* (13) *holds, then for all* $m, n, p \in \mathbf{N}$ *and* $f \in L^2$, *we have*

$$\langle [(A^m B^n)^p]^* (A^m B^n)^p f, f\rangle \geq \langle h_2^{(m+n)p} f, f\rangle \tag{16}$$

and

$$\langle [A^m B^n]^p (A^m B^n)^p]^* f, f\rangle \leq \langle h_2^{(m+n)p} f, f\rangle . \quad \square \tag{17}$$

REMARK. $h_2 \circ T_2 \leq h_2$ is not necessary in the proof of (16). Now we may easily prove our main result:

THEOREM 1. *If* (13) *holds,* $(A^m B^n)^p$ *is hyponormal for all* $m, n, p \in \mathbf{N}$.

PROOF. By first applying (16) and then applying (17) we have, for $m, n, p \in \mathbf{N}$ and $f \in L^2$,

$$\langle [(A^m B^n)^p]^* (A^m B^n)^p f, f\rangle \geq \langle h_2^{(m+n)p} f, f\rangle \geq \langle [A^m B^n]^p [(A^m B^n)^p]^* f, f\rangle ,$$

so that $\|(A^m B^n)^p f\| \geq \|[(A^m B^n)^p]^* f\|$. $\square$

The following corollaries are immediate.

COROLLARY 1. *With* A, B, h_1, *and* h_2 *as above, if* (13) *holds, then for each* $m, n \in \mathbf{N}$, $A^m B^n$ *is hyponormal.* $\square$

REMARK. This may be proved directly using Lemmas 2.3 and 2.4.

COROLLARY 2. *If* (13) *holds, then* $(AB)^p$ *is hyponormal for each* $p \in \mathbf{N}$.

REMARK. Observe that (13) implies a priori that both A and B are hyponormal. Actually, the conclusion of Corollary 2 is true under the weaker hypothesis that $h_i \circ T_i \leq h_j$, $i \neq j$, $i, j = 1, 2$. These hypotheses may be used to prove the inequalities

$$\langle [(AB)^p]^* (AB)^p f, f\rangle \geq \langle h_2^{2p} f, f\rangle \tag{18}$$

and

$$\langle (AB)^p [(AB)^p]^* f, f\rangle \leq \langle h_2^{2p} f, f\rangle , \tag{19}$$

from which Corollary 2 follows immediately.

COROLLARY 3. *With* T, h, *and* m *as in* §I, *if* $h \circ T \leq h$ *a.e.* m, *then* C_T^n *is hyponormal for each* $n \in \mathbf{N}$. $\square$

REMARK. Corollary 3 also follows directly from Lemmas 2.1 and 2.2.

III. An example. Halmos notes in [**1**] that it is not easy to find examples of hyponormal operators whose squares are not hyponormal. The simplest such example is due to Ito and Wong [**3**]; namely, that $U^* + 2U$, where U is the unilateral shift on $l^2(\mathbf{N})$, is hyponormal but its square is not. For a hyponormal composition operator whose square is not hyponormal, we have the following

EXAMPLE. Let $X = [0, \infty)$ be equipped with Lebesgue measure on the Borel sets. Set

$$\begin{aligned} Tx = x\chi_{[0,1)} &+ (10x/3 - 10/3)\chi_{(1,13/10]} + (5x/2 - 9/4)\chi_{(13/10,3/2]} \\ &+ (3x/10 - 9/20)\chi_{(3/2,29/6]} + (x/100 + 871/600)\chi_{(29/6,\infty)}. \end{aligned}$$

Then

$$h = dm \circ T^{-1}/dm = 139/30\chi_{[0,1)} + 2/5\chi_{(1,3/2]} + 100\chi_{(3/2,\infty)}.$$

($\chi_{[a,b)}$ stands for the characteristic function of $[a, b)$.) Using Lemma 15 of [**2**], it can be shown that C_T is hyponormal. If one considers $g(x) = \chi_{(1,13/10]}$, then a direct but tedious computation shows that $\langle C_T^{2*} C_T^2 g, g\rangle < \langle C_T^2 C_T^{2*} g, g\rangle$ so that C_T^2 is not hyponormal.

BIBLIOGRAPHY

1. P. R. Halmos, *A Hilbert space problem book*, Van Nostrand, Princeton, N. J., 1976.
2. D. J. Harrington and R. Whitley, *Seminormal composition operators*, J. Operator Theory **11** (1984), 125–135.
3. T. Ito and T. K. Wong, *Subnormality and quasinormality of Toeplitz operators*, Proc. Amer. Math. Soc. **34** (1972), 157–164.
4. A. Lambert, *Hyponormal composition operators*, Bull. London Math. Soc. **18** (1986), 395–400.
5. E. Nordgren, *Composition operators in Hilbert space*, Hilbert Space Operators (Proc. 1977), Lecture Notes in Math., vol. 693, Springer-Verlag, Berlin, 1978, pp. 37–63.

DEPARTMENT OF MATHEMATICAL SCIENCES, MEMPHIS STATE UNIVERSITY, MEMPHIS, TENNESSEE 38152