PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 102, Number 4, April 1988

CONVOLUTION OPERATORS ON GROUPS AND MULTIPLIER THEOREMS FOR HERMITE AND LAGUERRE EXPANSIONS

JOLANTA DŁUGOSZ

(Communicated by John B. Conway)

ABSTRACT. Using harmonic analysis on nilpotent Lie groups the following theorem is proved.

Let a sequence $\{a_{\mathbf{n}}\}$ be defined by a function $K \in C^N(\mathbf{R}_+)$ such that $\sup_{\lambda>0} |K^{(j)}(\lambda)\lambda^j| < \infty$, $j = 0, 1, \ldots, N$, for N sufficiently large, putting $a_{\mathbf{n}} = K(|\mathbf{n}| + m/2)$. Let $\varphi_{\mathbf{n}}$ be either Hermite or Laguerre functions. Then the operator

$$\sum_{\mathbf{n}} (f, \varphi_{\mathbf{n}})\varphi_{\mathbf{n}} \to \sum_{\mathbf{n}} a_{\mathbf{n}}(f, \varphi_{\mathbf{n}})\varphi_{\mathbf{n}}$$

is bounded on $L^p(\mathbf{R}^m)$ or $L^p(\mathbf{R}^m_+)$ respectively, $1 < p < \infty$.

0. In 1981 A. Hulanicki and J. W. Jenkins applied harmonic analysis on nilpotent Lie groups to obtain summability results for Hermite expansions [**12**]. The methods they initiated have been developed and used in a number of papers by A. Hulanicki and J. W. Jenkins [**13**] and others [**5**, **19**]. In the present paper we show that a simple combination of the so-called transference method of R. R. Coifman and G. Weiss with the above mentioned aspects of harmonic analysis on graded nilpotent Lie groups produces multiplier theorems for Laguerre and Hermite expansions.

After introductory §1, in §2 we prove a simple theorem concerning images by representations of operators which are functions of Rockland operators on graded nilpotent Lie groups. The proof is based on results of A. Hulanicki [**11**] and A. Hulanicki and E. M. Stein [**7**] and some unpublished remarks of A. Hulanicki.

§3 is devoted to multiplier theorems for Laguerre and Hermite expansions, some of which have been previously obtained [**6**, **19**] by much more complicated arguments.

1. Let G be an amenable group. Denote by $Cv^p(G)$ the algebra of operators bounded on $L^p(G)$ which commute with left translations and $|||T|||_p$ the operator norm of $T \in Cv^p(G)$. Let X be a σ-finite measure space and let a representation π of G act on $L^p(X)$. We call π uniformly bounded if there is a constant C such that $\|\pi(x)\| \le C$ for all x in G. A function $k \in L^1(G)$ defines a convolution operator on $L^p(G)$ by

$$f(x) \to f * k(x) = \int_G f(xy^{-1})k(y)\,dy. \tag{1.1}$$

Denote its norm by $|||k|||_p$.

Received by the editors November 6, 1986.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 43A80, 22E30; Secondary 33A65.

Key words and phrases. Multiplier, Laguerre expansions, Hermite expansions, convolution operators, Rockland operator.

The following theorem follows from the now classical papers of Herz [**9**, **10**].

THEOREM A. *If π is a uniformly bounded representation of an amenable group G on an $L^p(X)$ space, then for every $k \in L^1(G)$ the operator*

$$\pi(k)f = \int_G k(x)\pi(x)f\,dx, \qquad f \in L^p(X), \tag{1.2}$$

is bounded on $L^p(X)$ and $\|\pi(k)\| \leq C^2 |\!|\!|k|\!|\!|_p$, where C is a constant such that $\|\pi(x)\| \leq C$ for all $x \in G$.

In fact, under the assumption that k has compact support this theorem is due to Coifman and Weiss [**3**, **4**] (cf. also more recent papers of Anker [**1**, **2**]).

2. Now let G be a graded nilpotent Lie group. Let L be a positive Rockland operator on G that is a homogeneous, left-invariant differential operator such that for every nontrivial irreducible unitary representation π of G the operator $\pi(L)$ is injective on the space of C^∞ vectors. As such, L is hypoelliptic and essentially selfadjoint on $L^2(G)$ (cf. [**7**]).

Let $E(\lambda)$ be a spectral resolution of the identity on $L^2(G)$ corresponding to L. For $K \in L^\infty(\mathbb{R}_+)$ define the operator

$$T_K f = \int_0^\infty K(\lambda)\,dE(\lambda)f, \qquad f \in L^2(G), \tag{2.1}$$

which is in $Cv^2(G)$.

In [**11**] Hulanicki proved that if $K \in C^N(\mathbb{R}_+)$ satisfies conditions

$$\sup_{\lambda>0} |K^{(j)}(\lambda)(1+\lambda)^M| < \infty, \qquad j = 0, 1, \ldots, N, \tag{2.2}$$

where M, N are some constants ($M > N$), then the operator T_K defined by (2.1) is of the form

$$T_K f = f * k, \tag{2.3}$$

where k is in $L^1(G)$. He remarked also that a sufficient condition on K to imply the boundedness of T_K on all the spaces $L^p(G)$, $1 < p < \infty$, is: $K \in C^N(\mathbb{R}_+)$ for some constant N and

$$\sup_{\lambda>0} |K^{(j)}(\lambda)\lambda^j| < \infty, \qquad j = 0, 1, \ldots, N. \tag{2.4}$$

The proof of this fact is similar to the one given by Hulanicki and Stein (cf. [**7**, Chapter 6]).

Let π be a unitary representation of G on $L^2(X)$, where X is a measure space. For a function K satisfying (2.2) put

$$\pi(K)f = \pi(k)f = \int_G k(x)\pi(x)f\,dx, \qquad f \in L^2(X), \tag{2.5}$$

where $k \in L^1(G)$ is such that $T_K f = f * k$. Since the set of functions satisfying (2.2) is dense in $C_0(\mathbb{R}_+)$ we can extend (2.5) to a representation of the commutative

C^* algebra $C_0(\mathbb{R}_+)$. By a known theorem of Naimark [**17**, p. 246] there exists a spectral measure $E^\pi(\lambda)$ on $L^2(X)$ such that

$$\pi(T_K)f = \pi(K)f = \int_0^\infty K(\lambda)\,dE^\pi(\lambda)f, \qquad f \in L^2(X). \tag{2.6}$$

Now, Theorem A of §1 combined with results mentioned above give the following theorem on the boundedness of the above type operators in the $L^p(X)$ norm.

THEOREM 1. *Let π be a unitary representation of G on $L^2(X)$. Assume that there is a $p \in (1,\infty)$ such that π is uniformly bounded in the $L^p(X)$ norm. Let $K \in C^N(\mathbb{R}_+)$, where N is large enough to guarantee the differentiability needed both in* (2.2) *and* (2.4). *Define T_K by the formula* (2.1), *where $E(\lambda)$ is the spectral resolution of the identity on $L^2(G)$ corresponding to a positive Rockland operator. If the function K satisfies conditions* (2.4), *then the operator*

$$\pi(T_K)f = \int_0^\infty K(\lambda)\,dE^\pi(\lambda)f, \qquad f \in L^2 \cap L^p(X), \tag{2.7}$$

is bounded in the $L^p(X)$ norm.

PROOF. Since K satisfies conditions (2.4), the operator T_K is bounded on $L^p(G)$. If, additionally, K satisfies conditions (2.2), then T_K is a convolution operator by a function in $L^1(G)$ and Theorem 1 follows by virtue of Theorem A.

Now let K be any function satisfying the assumptions of the theorem. Take a function $\varphi \in C^\infty(\mathbb{R}_+)$ such that $\operatorname{supp}\varphi \subset [1/2, 2]$ and $\varphi(\lambda) = 1$ for $\lambda \in [3/4, 3/2]$ and consider the functions $K_\varepsilon(\lambda) = K(\lambda)\varphi_\varepsilon(\lambda)$, where $\varphi_\varepsilon(\lambda) = \varphi(\lambda^\varepsilon)$, $\varepsilon \in (0, \varepsilon_0]$. The functions $K_\varepsilon(\lambda)$ satisfy conditions (2.2) so the theorem holds for $K = K_\varepsilon$.

Observe also that $\varphi_\varepsilon^{(j)}(\lambda)\lambda^j$ is of the form $\sum_{m=1}^{j} P_m(\varepsilon)\lambda^{m\varepsilon}\varphi^{(m)}(\lambda^\varepsilon)$, where $P_m(\varepsilon)$ are polynomials. Therefore there is a constant C such that

$$\sup_{\lambda>0} |\varphi_\varepsilon^{(j)}(\lambda)\lambda^j| \leq C, \qquad j = 0, 1, \ldots, N,$$

for all $\varepsilon \in (0, \varepsilon_0]$. It follows that the corresponding convolution operators T_{φ_ε} defined by (2.1) are uniformly bounded on $L^p(G)$, that is $|||T_{\varphi_\varepsilon}|||_p \leq C_1$, where C_1 is independent of ε (cf. [**7**, Chapter 6]). Thus

$$|||T_{K_\varepsilon}|||_p \leq |||T_K|||_p |||T_{\varphi_\varepsilon}|||_p \leq C_1 |||T_K|||_p$$

and, again by Theorem A, there is a constant C_2 such that

$$\|\pi(T_{K_\varepsilon})\|_{(L^p(X), L^p(X))} \leq C_2 |||T_{K_\varepsilon}|||_p \leq C_1 C_2 |||T_K|||_p.$$

Moreover, since $\varphi_\varepsilon(\lambda) \to 1$ for all λ we have

$$\lim_{\varepsilon\to 0} \|\pi(T_{K_\varepsilon})f - \pi(T_K)f\|_{L^2(X)} = 0, \qquad f \in L^2 \cap L^p(X).$$

So we can choose a sequence $\pi(T_{K_{\varepsilon_n}})f$ tending to $\pi(T_K)f$ almost everywhere and using Fatou's lemma we get $\|\pi(T_K)f\|_{L^p(X)} \leq C_1 C_2 |||T_K|||_p \|f\|_{L^p(X)}$ which proves the theorem.

3. As corollaries we present multiplier theorems for classical Laguerre and Hermite expansions. To obtain them we consider the Heisenberg group and the spectral resolution $E(\lambda)$ corresponding to the sublaplacian.

Let H_m denote the $(2m+1)$-dimensional Heisenberg group. We write the elements of H_m as pairs (z,u), $z \in \mathbb{C}^m$, $u \in \mathbb{R}$, the multiplication law being

$$(z,u)(z',u') = \left(z + z', u + u' + \operatorname{Im} \sum_{j=1}^{m} z_j \overline{z'_j} \right).$$

Let X_j (Y_j), $j = 1,2,\ldots,m$, be the left-invariant vector fields on H_m corresponding to the one-parameter subgroups $(0,\ldots,0,t,0,\ldots,0)$ $((0,\ldots,0,it,0,\ldots,0))$ (t on the jth place). The homogeneous sublaplacian L on H_m is defined by $L = -\sum_{j=1}^{m}(X_j^2 + Y_j^2)$. Let $\Gamma = \{(0,n) \in H_m : n \in \mathbb{Z}\}$. On H_m/Γ consider the functions of the form

$$f(z_1,\ldots,z_m,u) = e^{2\pi i u} e^{-i\sum_{j=1}^{m}\alpha_j\theta_j} f_0(r_1,\ldots,r_m), \tag{3.1}$$

where $z_j = r_j e^{i\theta_j}$, $j = 1,\ldots,m$, and $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_m) \in \{0,1,2,\ldots\}^m = \mathbb{N}^m$.

Denote by $L^p_{\boldsymbol{\alpha}}(H_m/\Gamma)$ the space of the functions in $L^p(H_m/\Gamma)$ of the form (3.1). Let

$$\mathcal{L}_n^\alpha(w) = \left(\frac{n!}{\Gamma(n+\alpha+1)}\right)^{1/2} e^{-w/2} w^{\alpha/2} L_n^\alpha(w)$$

be the Laguerre function of type α, where

$$L_n^\alpha(w) = \sum_{j=0}^{n} \binom{n+\alpha}{n-j} \frac{(-w)j}{j!}$$

is the corresponding Laguerre polynomial. Let

$$\mathcal{L}_{\mathbf{n}}^{\boldsymbol{\alpha}}(2\pi\mathbf{r}^2) = \mathcal{L}_{n_1}^{\alpha_1}(2\pi r_1^2)\cdots\mathcal{L}_{n_m}^{\alpha_m}(2\pi r_m^2).$$

In [**5**] we proved the following.

LEMMA 1. *The operator L has a discrete spectrum on $L^2_{\boldsymbol{\alpha}}(H_m/\Gamma)$ with an orthonormal basis of eigenfunctions*

$$\varphi_{\mathbf{n}}^{\boldsymbol{\alpha}}(z,u) = 2^{m/2} e^{2\pi i u} e^{-i\sum_{j=1}^{m}\alpha_j\theta_j} \mathcal{L}_{\mathbf{n}}^{\boldsymbol{\alpha}}(2\pi\mathbf{r}^2), \qquad \mathbf{n} \in \mathbb{N}^m;$$

the corresponding eigenvalues being $8\pi(|\mathbf{n}| + m/2)$, where $|\mathbf{n}| = n_1 + \cdots + n_m$.

THEOREM 2 (cf. [**6**]). *Let K be a function in $C^N(\mathbb{R}_+)$, $N > m+3$, satisfying the conditions (2.4). For a function g in $L^p(\mathbb{R}^m_+)$ with the Laguerre expansion*

$$g \sim \sum_{\mathbf{n}\in\mathbb{N}^m} (g, \mathcal{L}_{\mathbf{n}}^{\boldsymbol{\alpha}})\mathcal{L}_{\mathbf{n}}^{\boldsymbol{\alpha}}$$

let $\tilde{T}_K g$ be defined by its Laguerre expansion

$$\tilde{T}_K g \sim \sum_{\mathbf{n}\in\mathbb{N}^m} K(|\mathbf{n}| + m/2)(g, \mathcal{L}_{\mathbf{n}}^{\boldsymbol{\alpha}})\mathcal{L}_{\mathbf{n}}^{\boldsymbol{\alpha}}.$$

Then the operator $\tilde{T}_K$ is a bounded operator on $L^p(\mathbb{R}^m_+)$, $1 < p < \infty$.

PROOF. Let $K_1(t) = K(t/8\pi)$. In Theorem 1 take for G the Heisenberg group H_m, for $E(\lambda)$ the spectral resolution of the identity on $L^2(H_m)$ corresponding to the sublaplacian and for π the left quasiregular representation of H_m on $L^p(H_m/\Gamma)$. As for the class of differentiability C^N of K needed in Theorem 1 the best N obtained for this case is $N > m+3$ (cf. [**12**, **14**, **15**]). We conclude that the operator $\pi(T_{K_1})$ is bounded on $L^p_{\boldsymbol{\alpha}}(H_m/\Gamma)$, $1 < p < \infty$. Moreover, by Lemma 1 $\pi(T_{K_1})\varphi$ has the expansion

$$\pi(T_{K_1})\varphi \sim \sum_{\mathbf{n}\in\mathbb{N}^m} K(|\mathbf{n}| + m/2)(\varphi, \varphi^{\boldsymbol{\alpha}}_{\mathbf{n}})\varphi^{\boldsymbol{\alpha}}_{\mathbf{n}}$$

for φ in $L^p_{\boldsymbol{\alpha}}(H_m/\Gamma)$.

For φ in $L^p_{\boldsymbol{\alpha}}(H_m/\Gamma)$ of the form (3.1) define $P\varphi \in L^p(\mathbb{R}^m_+)$ by $P\varphi(\mathbf{w}) = f_0(\sqrt{w_1/2\pi}, \dots, \sqrt{w_m/2\pi})$. Since $\|\varphi\|_{L^p_{\boldsymbol{\alpha}}(H_m/\Gamma)} = C_p\|P\varphi\|_{L^p(\mathbb{R}^m_+)}$, where C_p is a positive constant, the correspondence $\varphi \to P\varphi$ is a bijection. We have also $(\varphi, \varphi^{\boldsymbol{\alpha}}_{\mathbf{n}}) = 2^{-m/2}(P\varphi, \mathcal{L}^{\boldsymbol{\alpha}}_{\mathbf{n}})$. So $P(\pi(T_{K_1})\varphi) = \tilde{T}_K g$, where $\varphi = P^{-1}g$, and Theorem 2 follows.

THEOREM 3 (cf. [**19**, Theorem 1]). *Let K be as in Theorem 2. Let $\mathcal{H}_{\mathbf{n}}(\mathbf{w}) = \prod_{j=1}^m \mathcal{H}_{n_j}(w_j)$, $\mathbf{w} = (w_1, \dots, w_m) \in \mathbb{R}^m$, where $\mathcal{H}_k$ is the kth Hermite function. For a function $g \in L^p(\mathbb{R}^m)$ with the Hermite expansion*

$$g \sim \sum_{\mathbf{n}\in\mathbb{N}^m} (g, \mathcal{H}_{\mathbf{n}})\mathcal{H}_{\mathbf{n}}$$

let $\tilde{T}_K g$ be defined by its Hermite expansion

$$\tilde{T}_K g \sim \sum_{\mathbf{n}\in\mathbb{N}^m} K(|\mathbf{n}| + m/2)(g, \mathcal{H}_{\mathbf{n}})\mathcal{H}_{\mathbf{n}}.$$

Then the operator $\tilde{T}_K$ is bounded on $L^p(\mathbb{R}^m)$, $1 < p < \infty$.

PROOF. Take as π in Theorem 1 the Schrödinger representation π_μ, $\mu \neq 0$, of H_m,

$$\pi_\mu(\mathbf{x}, \mathbf{y}, u)f(\mathbf{w}) = e^{i\mu u}e^{2i\mu\langle 2\mathbf{w}-\mathbf{x},\mathbf{y}\rangle}f(\mathbf{w} - \mathbf{x}),$$

where $\mathbf{z} = \mathbf{x}+i\mathbf{y} \in \mathbb{C}^m$, $u \in \mathbb{R}$, $\mathbf{w} \in \mathbb{R}^m$ and $f \in L^p(\mathbb{R}^m)$. Let L be the sublaplacian on H_m. The closure of $d\pi_\mu(L)$ in $L^2(\mathbb{R}^m)$ is equal to the operator

$$H(\mu) = \sum_{N=0}^{\infty} (2N + m)|\mu|P^{(m)}_N(\mu),$$

where $P^{(m)}_N(\mu)$ is the orthgonal projection on the finite dimensional subspace of $L^2(\mathbb{R}^m)$ spanned by an orthonormal system

$$\mathcal{H}_{\mathbf{n},\mu}(\mathbf{w}) = (2|\mu|^{1/2})^{m/2}\prod_{j=1}^{m} \mathcal{H}_{n_j}(2|\mu|^{1/2}w_j), \qquad |\mathbf{n}| = N$$

(cf. [**8**]). A straightforward application of Theorem 1 for the case $\mu = 1/4$ and $K_1(\lambda) = K(2\lambda)$ gives the desired result.

REMARK. Let a sequence $\{n_j\}_{j=1}^{\infty} \subset \mathbb{N}$ be such that $n_{j+1}/n_j \geq q > 1$. Denote $\mathcal{F} = \{f \in L^p\colon (f, \varphi_n) = 0 \text{ for } n \neq n_j,\ j = 1, 2, \dots\}$, where φ_n, $n \in \mathbb{N}$, are either Hermite or Laguerre functions. Let $\{a_j\}_{j=1}^{\infty} \in l^\infty(\mathbb{C})$. The operator

$$\mathcal{F} \ni \sum_{j=1}^{\infty}(f, \varphi_{n_j})\varphi_{n_j} \to \sum_{j=1}^{\infty} a_j(f, \varphi_{n_j})\varphi_{n_j} \in \mathcal{F} \tag{3.2}$$

is bounded in the L^p norm both in the Hermite and Laguerre case. We think this must be known but we do not know the references.

PROOF. Take a function $\psi \in C^\infty(\mathbf{R})$ such that $\operatorname{supp}\psi \subset (1/q, q)$, $\psi(1) = 1$. Let $\psi_j(\lambda) = \psi(\lambda/n_j)$. Put $K(\lambda) = \sum_{j=1}^\infty a_j\psi_j(\lambda - m/2)$. We have $K(n_j + m/2) = a_j$, $j = 1, 2, \ldots$. It is easy to verify that K satisfies conditions (2.4). So the operator defined by (3.2) is bounded by virtue of Theorems 2 and 3.

ACKNOWLEDGMENT. The author is grateful to Andrzej Hulanicki for his help and remarks. In fact, the proof of Theorem 1 is due to him. The author is grateful also to Marek Bożejko and Tadeusz Pytlik for stimulating discussions and to Gero Fendler for pointing out Anker's results.

REFERENCES

1. J.-Ph. Anker, *Aspects de la p-induction en analyse harmonique*, Thèse de doctorat, Payot Lausanne, 1982.
2. ——, *Application de la p-induction en analyse harmonique*, Comment. Math. Helv. **58** (1983), 622–645.
3. R. R. Coifman and G. Weiss, *Operators associated with representations of amenable groups, singular integrals induces by ergodic flows, the rotation method and multipliers*, Studia Math. **57** (1973), 285–303.
4. ——, *Transference methods in analysis*, CBMS Regional Conf. Ser. in Math., no. 31, Amer. Math. Soc., Providence, R.I., 1977.
5. J. Długosz, *Almost everywhere convergence of some summability methods for Laguerre expansions*, Studia Math. **82** (1985), 199–209.
6. ——, *L^p multipliers for Laguerre expansions*, Colloq. Math. (to appear).
7. G. Folland and E. M. Stein, *Hardy spaces on homogeneous groups*, Princeton Univ. Press, Princeton, N.J., 1982.
8. D. Geller, *Fourier analysis on the Heisenberg group*, Ph.D. Thesis, Princeton Univ., Princeton, N.J., 1977.
9. C. Herz, *The theory of p-spaces with an application to convolution operators*, Trans. Amer. Math. Soc. **154** (1971), 69–82.
10. ——, *Harmonic synthesis for subgroups*, Ann. Inst. Fourier (Grenoble) **23** (1973), 91–123.
11. A. Hulanicki, *A functional calculus for Rockland operators on nilpotent Lie groups*, Studia Math. **78** (1984), 253–266.
12. A. Hulanicki and J. W. Jenkins, *Almost everywhere summability on nilmanifolds*, Trans. Amer. Math. Soc. **278** (1983), 703–715.
13. ——, *Nilpotent Lie groups and summability of eigenfunction expansions of Schrödinger operators*, Studia Math. **80** (1984), 235–244.
14. A. Hulanicki and E. M. Stein, *Marcinkiewicz multiplier theorem for stratified groups* (manuscript).
15. G. Mauceri, *Zonal multipliers on the Heisenberg group*, Pacific J. Math. **95** (1981), 143–159.
16. G. Mauceri and L. de Michele, *L^p multipliers on the Heisenberg group*, Michigan Math. J. **26** (1979), 361–371.
17. M. A. Naimark, *Normed rings*, Noordhoff, Groningen, 1959.
18. E. M. Stein, *Topics in harmonic analysis related to the Littlewood-Paley theory*, Princeton Univ. Press, Princeton, N.J., 1970.
19. K. Stempak, *Multipliers for eigenfunction expansions of some Schrödinger operators*, Proc. Amer. Math. Soc. **93** (1985), 477–482.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WROCŁAW, PL. GRUNWALDZKI 7A, 50-370 WROCŁAW, POLAND