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SPECTRAL MANIFOLDS OF BOUNDED S-DECOMPOSABLE OPERATORS

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ABSTRACT. We prove some properties of spectral manifolds of a bounded S-decomposable operator on a complex Banach space. Also, we prove a new characterization of S-decomposability.

Introduction. In this paper, we consider a bounded linear operator T on a complex Banach space X. $\mathrm{Lat}(T)$ is the lattice of all invariant subspaces of T. For $Y \in \mathrm{Lat}(T)$, $T|Y$ is the restriction of T to Y and T/Y is the induced operator on X/Y by T. In the theory of decomposable operator, the spectral manifold $X_T(F)$ has played an important role. For example, if T is decomposable, then (#) $X_T(F)$ is closed, $\sigma(T|X_T(F)) \subset F$ and $\sigma(T/X_T(F)) \subset \overline{\mathbf{C} \setminus F}$ for all closed sets $F \subset \mathbf{C}$. Conversely if there exists $Y \in \mathrm{Lat}(T)$ such that $\sigma(T|Y) \subset F$ and $\sigma(T/Y) \subset \overline{\mathbf{C} \setminus F}$ for every closed set $F \subset \mathbf{C}$, then T is decomposable. Hence condition (#) is an equivalent condition for T to be decomposable. In this paper, we investigate the spectral manifolds $X_T(F)$, $\overline{X_T(G)}$ and $X_T(\overline{G \cap \sigma(T)})$ of an S-decomposable operator T where $F \subset \mathbf{C}$ is closed and $G \subset \mathbf{C}$ is open. We first investigate the spectrum of the restriction and the induced operator (Theorems 1, 3, 4), and then we prove that the above equivalent condition (#) can be extended to the case in which T is S-decomposable (Theorem 5).

$Y \in \mathrm{Lat}(T)$ is a spectral maximal space of T if Y includes all $Z \in \mathrm{Lat}(T)$ with $\sigma(T|Z) \subset \sigma(T|Y)$. $\mathrm{SM}(T)$ is the family of all spectral maximal spaces of T. Let $S \subset \mathbf{C}$ be a closed set. A family $\{G_1, G_2, \ldots, G_n; G_0\}$ of open sets is an S-covering of $\sigma(T)$ if $G_1 \cup G_2 \cup \cdots \cup G_n \cup G_0 \supset \sigma(T) \cup S$ and $\overline{G}_i \cap S = \varnothing$ for $i = 1, 2, \ldots, n$. T is S-decomposable if, for every S-covering of $\sigma(T)$, there exists a system of spectral maximal spaces $\{X_1, X_2, \ldots, X_n, X_0\}$ such that $X = X_1 + X_2 + \cdots + X_n + X_0$ and $\sigma(T|X_i) \subset G_i$ for $i = 0, 1, \ldots, n$. For a closed set $F \subset \mathbf{C}$, $X_T(F) = \{x \in X |$ there exists an analytic function $f: \mathbf{C} \setminus F \to X$ such that $(z - T)f(z) = x$ on $\mathbf{C} \setminus F\}$. Moreover, for any set $E \subset \mathbf{C}$, one has $X_T(E) = \bigcup\{X_T(F) | F \subset E$ and F is closed$\}$. We remark that this definition of the spectral manifold is due to M. Radjabalipour [8]. Let $\Omega \subset \mathbf{C}$ be an open set. $Y \in \mathrm{Lat}(T)$ is Ω-analytically invariant under T if $(z - T)f(z) \in Y$ on D implies $f(z) \in Y$ on D for every open set with $D \subset \Omega$ and for every analytic function $f: D \to X$. T has Ω-svep if there exists no nonzero analytic function $g: D \to X$ such that $(z - T)g(z) = 0$ on D for every open set D with $D \subset \Omega$.

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W. Shengwang and L. Guangyu [**9**] proved that, for $Y \in \mathrm{Lat}(T)$, T/Y has Ω-svep iff Y is Ω-analytically invariant under T. They also proved that T is S-decomposable iff T^* is S-decomposable, where T^* is the dual operator of T. J. K. Finch [**5**] proved that if T has Ω-svep and $z - T$ is surjective for some $z \in \Omega$, then $z \in \rho(T)$. (He actually proved the case $\Omega = \mathbf{C}$, but the proof of this result is similar.) I. Bacalu [**1**] proved that if T is S-decomposable, then T has S^c-svep. We use the relation $\sigma(T) \subset \sigma(T|T) \cup \sigma(T/Y)$ for $Y \in \mathrm{Lat}(T)$ repeatedly. E^i is the interior of E and $\overline{E}$ is the (norm) closure of E.

Some further notations are $\mathrm{Lat}(T)$ for the lattice of all invariant subspaces under T, and $\mathrm{SM}(T)$ for the family of all spectral maximal spaces of T. The abbreviation svep stands for the single valued extension property.

Main results.

THEOREM 1. *Let T be S-decomposable.*

(1) *If F is a closed set with $S \subset F$, then $X_T(F) \in \mathrm{SM}(T)$, $\overline{(F^i \setminus S) \cap \sigma(T)} \subset \sigma(T|X_T(F)) \subset F \cap \sigma(T)$ and $\overline{\sigma(T) \setminus F} \subset \sigma(T/X_T(F)) \subset \overline{\sigma(T) \setminus F} \cup S$.*

(2) *If G is an open set with $S \subset G$, then $X_T(\overline{G}) \in \mathrm{SM}(T)$, $\overline{G \cap \sigma(T)} \subset \sigma(T|X_T(\overline{G})) \subset \overline{G} \cap \sigma(T)$ and $\sigma(T/X_T(\overline{G})) = \overline{\sigma(T) \setminus \overline{G}}$.*

(3) *If F is a closed set with $S \cap F = \varnothing$, then $X_T(F) \in \mathrm{SM}(T)$, $\overline{F^i \cap \sigma(T)} \subset \sigma(T|X_T(F)) \subset F \cap \sigma(T)$ and $\sigma(T/X_T(F)) = \overline{\sigma(T) \setminus F}$.*

For the proof of Theorem 1 we need the following Lemma.

LEMMA. *Let T be S-decomposable. If F is a closed set with $S \subset F$ or $S \cap F = \varnothing$, then $X_T(F) \in \mathrm{SM}(T)$, $\sigma(T|X_T(F)) \subset F$ and $X_T(F)$ is S^c-analytically invariant under T.*

PROOF. It is proved that $X_T(F) \in \mathrm{SM}(T)$ and $\sigma(T|X_T(F)) \subset F$ by [**1**] (the case $S \subset F$) and [**6**] (the case $S \cap F = \varnothing$). Let D be an open set with $D \cap S = \varnothing$ and $f\colon D \to X$ be an analytic function such that $(z - T)f(z) \in X_T(F)$ on D. We may assume D is an open ball. If $D \subset \sigma(T|X_T(F))$, then $f(z) \in X_T(F)$ on D because $X_T(F)$ is T-absorbent by [**3**, Theorem 1.3.7]. If $D \cap \rho(T|X_T(F)) \neq \varnothing$, then $(z-T)(z-T|X_T(F))^{-1}(z-T)f(z) = (z-T)f(z)$ on $D \cap \rho(T|X_T(F))$. Since T has S^c-svep by [**1**], we have $f(z) = (z - T|X_T(F))^{-1}(z - T)f(z) \in X_T(F)$ on $D \cap \rho(T|X_T(F))$, hence on D.

PROOF OF THEOREM 1. (1) Since $X_T(F)$ is hyperinvariant under T, $\sigma(T|X_T(F)) \subset F \cap \sigma(T)$ by the Lemma. We prove $\sigma(T/X_T(F)) \subset \overline{\sigma(T) \setminus F} \cup S$. Let $z \notin \overline{\sigma(T) \setminus F} \cup S$. We have only to prove $z - T^F$ is surjective by the Lemma, where $T^F = T/X_T(F)$. Since there exist open sets G_1 and G_0 such that $z \in G_1$, $\overline{G}_1 \cap (\overline{\sigma(T) \setminus F} \cup S) = \varnothing$, $z \notin \overline{G}_0$ and $\{G_1; G_0\}$ is an S-covering of $\sigma(T)$, there exist $X_1, X_0 \in \mathrm{SM}(T)$ such that $X = X_1 + X_0$ and $\sigma(T|X_i) \subset G_i$ for $i = 1, 0$. Let $x \in X$ be given. Then we can write $x = x_1 + x_0$ for some $x_i \in X_i$ for $i = 1, 0$. Then there exists $y \in X_0$ with $(z-T)y = x_0$. Since $X_1 \subset X_T(\overline{G}_1) = X_T(\overline{G}_1 \cap \sigma(T)) \subset X_T(F)$, we have $(z - T^F)\hat{y} = \hat{x}_0 = \hat{x}_1 + \hat{x}_0 = \hat{x}$ where $\hat{x} \in X/X_T(F)$ is the coset of x. Thus $z - T^F$ is surjective. Hence

$$\sigma(T|X_T(F)) \supset \sigma(T) \setminus \sigma(T/X_T(F)) \supset \sigma(T) \cap (\overline{\sigma(T) \setminus F} \cup S) \supset \sigma(T) \cap (F^i \setminus S),$$

and hence $\sigma(T|X_T(F)) \supset \overline{\sigma(T) \cap (F^i \setminus S)}$. Also

$$\sigma(T/X_T(F)) \supset \sigma(T) \setminus \sigma(T|X_T(F)) \supset \sigma(T) \setminus F,$$

and hence $\sigma(T|X_T(F)) \supset \overline{\sigma(T) \setminus F}$.

(2) We have $X_T(\overline{G}) \in \mathrm{SM}(T)$, $\sigma(T|X_T(\overline{G})) \subset \overline{G} \cap \sigma(T)$, $\overline{\sigma(T) \setminus \overline{G}} \subset \sigma(T/X_T(\overline{G})) \subset \overline{\sigma(T) \setminus \overline{G}} \cup S$ by [1]. [8, Proposition 1.4] proved $\sigma(T/X_T(\overline{G})) \subset \sigma(T) \setminus G$. Hence $\sigma(T|X_T(\overline{G})) \subset (\sigma(T) \setminus G) \cap (\overline{\sigma(T) \setminus \overline{G}} \cup S) = \overline{\sigma(T) \setminus \overline{G}}$, and hence $\sigma(T/X_T(\overline{G})) = \overline{\sigma(T) \setminus \overline{G}}$. Hence $\sigma(T|X_T(\overline{G})) \supset \sigma(T) \setminus \overline{\sigma(T) \setminus \overline{G}} \supset \sigma(T) \cap G$, and hence $\sigma(T|X_T(\overline{G})) \supset \overline{\sigma(T) \cap G}$.

(3) We prove $\sigma(T/X_T(F)) \subset \overline{\sigma(T) \setminus F}$. Let $z \notin \overline{\sigma(T) \setminus F}$. We have only to prove that $z - T^F$ is surjective by the Lemma. Since there exist open sets G_1 and G_0 such that $z \in G_1$, $\overline{G}_1 \cap \overline{\sigma(T) \setminus F} = \varnothing$, $z \notin \overline{G}_0$ and $\{G_1; G_0\}$ is an S-covering of $\sigma(T)$, there exist $X_1, X_0 \in \mathrm{SM}(T)$ such that $X = X_1 + X_0$ and $\sigma(T|X_i) \subset G_i$ for $i = 1, 0$. Let $x \in X$ be given. Then we can write $x = x_1 + x_0$ for some $x_i \in X_i$ for $i = 1, 0$. Hence there exists $y \in X_0$ with $(z - T)y = x_0$. Since $X_1 \subset X_T(\overline{G}_1) = X_T(\overline{G}_1 \cap \sigma(T)) \subset X_T(F)$, we have $(z - T^F)\hat{y} = \hat{x}_0 = \hat{x}_1 + \hat{x}_0 = \hat{x}$. Thus $z - T^F$ is surjective.Hence

$$\sigma(T|X_T(F)) \supset \sigma(T) \setminus \sigma(T|X_T(F)) \supset \sigma(T) \cap F^i,$$

and hence $\sigma(T|X_T(F)) \supset \overline{\sigma(T) \cap F^i}$. Also

$$\sigma(T/X_T(F)) \supset \sigma(T) \setminus \sigma(T|X_T(F)) \supset \sigma(T) \setminus F,$$

and hence $\sigma(T/X_T(F)) = \overline{\sigma(T) \setminus F}$.

REMARK. The idea of Theorem 1 is inspired by [4], which proved $\sigma(T/X_T(F)) = \overline{\sigma(T) \setminus F}$ for the strongly decomposable operator T.

THEOREM 2. *Let T be S-decomposable. If G is an open set with $S \subset G$ or $S \cap \overline{G} = \varnothing$, then $X_T(G)^\perp = X^*_{T^*}(G^c)$.*

PROOF. [10, Proposition 2.9] proved the case $S \cap \overline{G} = \varnothing$. We prove the case $S \subset G$. We may assume $S \subset \sigma(T)$. We prove first $X_T(G)^\perp \subset X^*_{T^*}(G^c)$. Let $x^* \in X_T(G)^\perp$. We write $G = \bigcup G_\alpha$ where G_α is a component of G. Since S is compact, we may assume that $\{\alpha | G_\alpha \cap S = \varnothing\}$ is finite. Let $S_\alpha = S \cap G_\alpha$. Then $S = \bigcup S_\alpha$ and S_α is closed. Hence there exist open sets G'_α, G''_α such that $S_\alpha \subset G'_\alpha$, $\overline{G'_\alpha} \subset G''_\alpha$ and $\overline{G''_\alpha} \subset G_\alpha$. Let $G_0 = \bigcup G''_\alpha$ and $G_1 = \bigcap (\overline{G'_\alpha})^c$. Since $\{G_1; G_0\}$ is an S-covering of $\sigma(T)$, there exist $X_1, X_0 \in \mathrm{SM}(T)$ such that $X = X_1 + X_0$ and $\sigma(T|X_i) \subset G_i$ for $i = 1, 0$. Let $x \in X$ be given. Then $x = x_1 + x_0$ for some $x_i \in X_i$ for $i = 1, 0$. We define

$$(x^*_{T^*}(z))(z) = x^*((z - T|X_1)^{-1} x_1) \quad \text{for } z \in \bigcup G'_\alpha.$$

We prove $x^*_{T^*}(z) \in X^*$. If $x = x_1 + x_0 = x'_1 + x'_0$ for some $x'_i \in X_i$ for $i = 1, 0$, then

$$\begin{aligned} x_1 - x'_1 = x'_0 - x_0 &\in X_1 \cap X_0 \subset X_T(\overline{G}_1) \cap X_T(\overline{G}_0) \\ &= X_T(\overline{G}_1 \cap \overline{G}_0) \subset X_T(G) \end{aligned}$$

because T has S^c-svep by [1] and $S \subset G_0$. Hence $x^*((z - T|X_1)^{-1}(x_1 - x'_1)) = 0$, and hence $x^*_{T^*}(z)$ is well defined. By the Banach theorem there exists $M > 0$ such that $\|x_1\| < M\|x\|$ for all $x \in X$. Hence

$$|(x^*_{T^*}(z))(x)| \le |x^*((z - T|X_1)^{-1}x_1)|$$
$$\le \|x^*\|\,\|(z - T|X_1)^{-1}\|M\|x\|,$$

and hence $x^*_{T^*}(z) \in X^*$. Then $x^*_{T^*}(z)$ is analytic on $\bigcup G'_\alpha$ and

$$((z - T^*)x^*_{T^*}(z))(x) = x^*_{T^*}(z)((z - T)x) = x^*(x_1)$$
$$= x^*(x_1 + x_0) = x^*(x)$$

because $X_0 \subset X_T(G_0) \subset X_T(G)$. Hence $(z - T^*)x^*_{T^*}(z) = x^*$ on $\bigcup G'_\alpha$ and hence $x^* \in X^*_{T^*}((\bigcup G'_\alpha)^c)$. Since G'_α is any open set with $S_\alpha \subset G'_\alpha \subset G_\alpha$ and T has S^c-svep, we have $x^* \in X^*_{T^*}(G^c)$, and hence $X_T(G)^\perp \subset X^*_{T^*}(G^c)$. The proof of the converse inclusion is easy. Thus $X_T(G)^\perp = X^*_{T^*}(G^c)$.

By the relations $\sigma(T) = \sigma(T^*)$, $(X/Y)^* = Y^\perp$ and $X^*/Y^\perp = Y^*$ for $Y \subset X$, we have the following theorem by Theorem 1.

THEOREM 3. *Let T be S-decomposable. If G is an open set with $S \subset G$ or $S \cap \overline{G} = \varnothing$, then $\sigma(T|\overline{X_T(G)}) = \overline{G \cap \sigma(T)}$ and $\overline{\sigma(T) \setminus \overline{G \cap \sigma(T)}} \subset \sigma(T|\overline{X_T(G)}) \subset \sigma(T) \setminus G$.*

It is easy to construct a decomposable operator which satisfies $\overline{F^i \cap \sigma(T)} \subsetneq \sigma(T|X_T(F)) \subsetneq F \cap \sigma(T)$ for some closed set $F \subset \overline{\mathbf{C}}$. We consider $X_T(\overline{G \cap \sigma(T)})$ where $G \subset \overline{\mathbf{C}}$ is open. It is clear that

$$\overline{X_T(G)} = \overline{X_T(G \cap \sigma(T))} \subset X_T(\overline{G \cap \sigma(T)}) \subset X_T(\overline{G}).$$

We prove that $X_T(\overline{G \cap \sigma(T)})$ has an interesting property.

THEOREM 4. *Let T be S-decomposable. If G is an open set with $S \subset G$ or $S \cap \overline{G} = \varnothing$, then $X_T(\overline{G \cap \sigma(T)}) \in \mathrm{SM}(T)$,*

$$\sigma(T|X_T(\overline{G \cap \sigma(T)})) = \overline{G \cap \sigma(T)},$$
$$\sigma(T/X_T(\overline{G \cap \sigma(T)})) = \overline{\sigma(T) \setminus \overline{G \cap \sigma(T)}}.$$

PROOF. We prove first the case $S \cap \overline{G} = \varnothing$. By Theorem 1, we have only to prove $\overline{G \cap \sigma(T)} \subset \sigma(T|X_T(\overline{G \cap \sigma(T)}))$. Let $z \in G \cap \sigma(T)$ be given. Then there exist open sets G_1, G_0 such that $z \in G_1$, $\overline{G}_1 \subset G$, $z \notin \overline{G}_0$ and $\{G_1; G_0\}$ is an S-covering of $\sigma(T)$. Hence there exist open sets G_1, G_0 such that $z \in G_1$, $\overline{G}_1 \subset G$, $z \notin G_0$ and $\{G_1; G_0\}$ is an S-covering of $\sigma(T)$. Hence there exist $X_1, X_0 \in \mathrm{SM}(T)$ such that $X = X_1 + X_0$ and $\sigma(T|X_i) \subset G_i$ for $i = 1, 0$. Then

$$X_1 \subset X_T(\overline{G}_1) \subset X_T(G) = X_T(G \cap \sigma(T)) \subset X_T(\overline{G \cap \sigma(T)}),$$

and $X_0 \subset X_T(\overline{G}_0)$. Hence

$$X = X_T(\overline{G \cap \sigma(T)}) + X_T(\overline{G}_0).$$

Then it is easy to prove

$$\sigma(T) \subset (T|X_T(\overline{G \cap \sigma(T)})) \cup \sigma(T|X_T(\overline{G}_0)) \cup S.$$

Since $z \in \sigma(T)$, $z \notin \sigma(T|X_T(G_0))$ and $z \notin S$, we have $z \in \sigma(T|X_T(\overline{G \cap \sigma(T)}))$. Hence $G \cap \sigma(T) \subset \sigma(T|X_T(\overline{G \cap \sigma(T)}))$, and $\overline{G \cap \sigma(T)} \subset \sigma(T|X_T(\overline{G \cap \sigma(T)}))$.

We prove the case $S \subset G$. By Theorem 1(1), $X_T(\overline{G \cap \sigma(T)}) \in \mathrm{SM}(T)$ and $\sigma(T|X_T(\overline{G \cap \sigma(T)})) \subset \overline{G \cap \sigma(T)}$. We prove $\sigma(T/X_T(\overline{G \cap \sigma(T)})) \subset G^c$. Let $z \in G$ be given. Then there exist open sets G_1, G_0 such that $z \in G_0$, $S \subset G_0$, $\overline{G}_0 \subset G$, $z \notin \overline{G}_1$ and $\{G_1; G_0\}$ is an S-covering of $\sigma(T)$. Hence there exist $X_1, X_0 \in \mathrm{SM}(T)$ such that $X = X_1 + X_0$ and $\sigma(T|X_i) \subset G_i$ for $i = 1, 0$. Since

$$X_0 \subset X_T(\overline{G}_0) \subset X_T(G) = X_T(G \cap \sigma(T)) \subset X_T(\overline{\sigma(T) \cap G}),$$

we have $X = X_T(\overline{G \cap \sigma(T)}) + X_T(\overline{G}_1)$. Since T has S^c-svep and $X_T(\overline{G}_1)$ is closed, we have

$$\begin{aligned} X/X_T(\overline{G \cap \sigma(T)}) &= X_T(\overline{G}_1)/(X_T(\overline{G \cap \sigma(T)}) \cap X_T(\overline{G}_1)) \\ &= X_T(\overline{G}_1)/X_T(\overline{G \cap \sigma(T)} \cap \overline{G}_1). \end{aligned}$$

Then

$$\sigma(T/X_T(\overline{G \cap \sigma(T)})) \subset \sigma(T|X_T(\overline{G}_1)) \cup \sigma(T|X_T(\overline{G \cap \sigma(T)}) \subset \overline{G}_1,$$

and hence $z \in \rho(T/X_T(\overline{G \cap \sigma(T)}))$. Hence, by Theorem 1(1),

$$\sigma(T/X_T(\overline{G \cap \sigma(T)})) \subset G^c \cap \overline{\{\sigma(T) \setminus \overline{G \cap \sigma(T)} \cup S\}} = \overline{\sigma(T) \setminus \overline{G \cap \sigma(T)}},$$

and hence $\sigma(T/X_T(\overline{G \cap \sigma(T)})) = \overline{\sigma(T) \setminus \overline{G \cap \sigma(T)}}$. Hence

$$\sigma(T|X_T(\overline{G \cap \sigma(T)})) \supset \sigma(T) \setminus \sigma(T/X_T(\overline{G \cap \sigma(T)})) \supset \overline{G \cap \sigma(T)},$$

and hence $\sigma(T|X_T(\overline{G \cap \sigma(T)})) = \overline{G \cap \sigma(T)}$.

REMARK. Hence $\overline{G \cap \sigma(T)}$ is a set spectrum of T (cf. [**3**]). Theorems 1, 3 and 4 imply that if T is S-decomposable, then the mappings $G \to X_T(\overline{G})$, $\overline{X_T(G)}$, $X_T(\overline{G \cap \sigma(T)}))$ are, we may say, "S-spectral resolvents" (cf. [**2**]). The following theorem proves that the converse implication is also true.

THEOREM 5. (1) *If, for every open set with $S \subset G$, there exists $Y \in \mathrm{Lat}(T)$ such that $\sigma(T|Y) \subset \overline{G}$ and $\sigma(T/Y) \subset G^c \cup S$, then T is S-decomposable.*

(2) *If, for every open set G with $S \cap \overline{G} = \varnothing$, there exists $Y \in \mathrm{Lat}(T)$ such taht $\sigma(T|Y) \subset \overline{G} \cup S$ and $\sigma(T/Y) \subset G^c$, then T is S-decomposable.*

PROOF. [**6**] proved (1). We prove (2) by using (1). Let G be an open set with $S \subset G$. Then $G_1 = \overline{G}^c$ is open and $\overline{G}_1 \cap S = \varnothing$. Hence there exists $Y \in \mathrm{Lat}(T)$ such that $\sigma(T|Y) \subset \overline{G}_1 \cup S$ and $\sigma(T/Y) \subset G_1^c$. Then

$$\sigma(T^*/Y^\perp) = \sigma((T|Y)^*) = \sigma(T|Y) \subset G_1 \cup S \subset G^c \cup S,$$

and

$$\sigma(T^*|Y^\perp) = \sigma((T/Y)^*) = \sigma(T/Y) = \overline{G}_1^c = \overline{G}.$$

Hence T^* is S-decomposable by (1). Hence T is S-decomposable by [**9**].

REMARK. [**9**] proved (2) under the conditions that $\sigma(T|Y) \subset \overline{G}$ and $\sigma(T/Y) \subset G^c \cap \sigma(T)$.

THEOREM 6. *Let T be S-decomposable. If F is a closed set with $S \subset F$ or $S \cap F = \varnothing$, then $JX_T(F) = JX \cap X^{**}_{T^{**}}(F)$ where $J: X \to X^{**}$ is the canonical embedding.*

PROOF. [**9**] proved the case $S \subset F$. We prove the case $S \cap F = \varnothing$. Let $Jx \in X^{**}_{T^{**}}(F)$ be given. Then there exists an analytic function $f: F^c \to X^{**}$ such that $(z - T^{**})f(z) = Jx$ on F^c. We write $F^c = \bigcup G_\alpha$ where G_α is a component of F^c. Since JX is S^c-analytically invariant under T^{**} by [**9**, Theorem 2.2] and $G_\alpha \cap S^c = \emptyset$, we have $f(z) \in JX$ on $G_\alpha \cap S^c$, hence on G_α. This implies $J^{-1}f(z) \in X$ exist for all $z \in \bigcup G_\alpha = F^c$. Hence $(z - T)J^{-1}f(z) = x$ on F^c, and hence $x \in X_T(F)$. Thus $JX \cap X^{**}_{T^{**}}(F) \subset JX_T(F)$. The proof of the converse inclusion is easy, hence the proof is finished.

THEOREM 7. *Let T be S-decomposable. If F is a closed set with $S \subset F$ or $S \cap F = \emptyset$, then $X_T(F)^\perp =$ the weak $*$-closure of $X^*_{T^*}(F^c)$.*

PROOF. We remark T^* is S-decomposable by [**10**]. Let $f \in X_T(F)^\perp$ and $f \notin$ the weak $*$-closure of $X^*_{T^*}(F^c)$. Then there exists $x \in X$ such that $x \perp X^*_{T^*}(F^c)$ and $f(x) = 1$. Hence $Jx \in X^*_{T^*}(F^c)^\perp = X^{**}_{T^{**}}(F)$ by Theorem 2, and $Jx \in JX \cap X^{**}_{T^{**}}(F) = JX_T(F)$. Hence $x \in X_T(F)$. This is a contradiction. Hence $X_T(F)^\perp \subset$ the weak $*$-closure of $X^*_{T^*}(F^c)$. The proof of the converse inclusion is easy, hence the proof is finished.

REMARK. If T is S-decomposable and F is a closed set with $S \subset F$ or $S \cap F = \emptyset$, then ${}^\perp X^*_{T^*}(F) = \{x \in X | x \perp X^*_{T^*}(F)\} = {}^\perp(X_T(F^c)^\perp) = \overline{X_T(F^c)}$ and $X^*_{T^*}(F)^\perp = X_T(F^c)^{\perp\perp} =$ the weak $*$-closure of $X_T(F^c)$ in X^{**}.

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