

SPECTRAL MANIFOLDS OF BOUNDED S -DECOMPOSABLE OPERATORS

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(Communicated by John B. Conway)

ABSTRACT. We prove some properties of spectral manifolds of a bounded S -decomposable operator on a complex Banach space. Also, we prove a new characterization of S -decomposability.

Introduction. In this paper, we consider a bounded linear operator T on a complex Banach space X . $\text{Lat}(T)$ is the lattice of all invariant subspaces of T . For $Y \in \text{Lat}(T)$, $T|Y$ is the restriction of T to Y and T/Y is the induced operator on X/Y by T . In the theory of decomposable operator, the spectral manifold $X_T(F)$ has played an important role. For example, if T is decomposable, then (#) $X_T(F)$ is closed, $\sigma(T|X_T(F)) \subset F$ and $\sigma(T/X_T(F)) \subset \overline{\mathbf{C}} \setminus \overline{F}$ for all closed sets $F \subset \mathbf{C}$. Conversely if there exists $Y \in \text{Lat}(T)$ such that $\sigma(T|Y) \subset F$ and $\sigma(T/Y) \subset \overline{\mathbf{C}} \setminus \overline{F}$ for every closed set $F \subset \mathbf{C}$, then T is decomposable. Hence condition (#) is an equivalent condition for T to be decomposable. In this paper, we investigate the spectral manifolds $X_T(F)$, $\overline{X_T(G)}$ and $X_T(\overline{G \cap \sigma(T)})$ of an S -decomposable operator T where $F \subset \mathbf{C}$ is closed and $G \subset \mathbf{C}$ is open. We first investigate the spectrum of the restriction and the induced operator (Theorems 1, 3, 4), and then we prove that the above equivalent condition (#) can be extended to the case in which T is S -decomposable (Theorem 5).

$Y \in \text{Lat}(T)$ is a spectral maximal space of T if Y includes all $Z \in \text{Lat}(T)$ with $\sigma(T|Z) \subset \sigma(T|Y)$. $\text{SM}(T)$ is the family of all spectral maximal spaces of T . Let $S \subset \mathbf{C}$ be a closed set. A family $\{G_1, G_2, \dots, G_n; G_0\}$ of open sets is an S -covering of $\sigma(T)$ if $G_1 \cup G_2 \cup \dots \cup G_n \cup G_0 \supset \sigma(T) \cup S$ and $\overline{G_i} \cap S = \emptyset$ for $i = 1, 2, \dots, n$. T is S -decomposable if, for every S -covering of $\sigma(T)$, there exists a system of spectral maximal spaces $\{X_1, X_2, \dots, X_n, X_0\}$ such that $X = X_1 + X_2 + \dots + X_n + X_0$ and $\sigma(T|X_i) \subset G_i$ for $i = 0, 1, \dots, n$. For a closed set $F \subset \mathbf{C}$, $X_T(F) = \{x \in X \mid \text{there exists an analytic function } f: \mathbf{C} \setminus F \rightarrow X \text{ such that } (z - T)f(z) = x \text{ on } \mathbf{C} \setminus F\}$. Moreover, for any set $E \subset \mathbf{C}$, one has $X_T(E) = \bigcup \{X_T(F) \mid F \subset E \text{ and } F \text{ is closed}\}$. We remark that this definition of the spectral manifold is due to M. Radjabalipour [8]. Let $\Omega \subset \mathbf{C}$ be an open set. $Y \in \text{Lat}(T)$ is Ω -analytically invariant under T if $(z - T)f(z) \in Y$ on D implies $f(z) \in Y$ on D for every open set with $D \subset \Omega$ and for every analytic function $f: D \rightarrow X$. T has Ω -svsp if there exists no nonzero analytic function $g: D \rightarrow X$ such that $(z - T)g(z) = 0$ on D for every open set D with $D \subset \Omega$.

Received by the editors August 25, 1986 and, in revised form, December 1, 1986.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 47B40; Secondary 47A05.

Key words and phrases. S -decomposable operator, spectral manifold.

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W. Shengwang and L. Guangyu [9] proved that, for $Y \in \text{Lat}(T)$, T/Y has Ω -svep iff Y is Ω -analytically invariant under T . They also proved that T is S -decomposable iff T^* is S -decomposable, where T^* is the dual operator of T . J. K. Finch [5] proved that if T has Ω -svep and $z - T$ is surjective for some $z \in \Omega$, then $z \in \rho(T)$. (He actually proved the case $\Omega = \mathbb{C}$, but the proof of this result is similar.) I. Bacalu [1] proved that if T is S -decomposable, then T has S^c -svep. We use the relation $\sigma(T) \subset \sigma(T|T) \cup \sigma(T/Y)$ for $Y \in \text{Lat}(T)$ repeatedly. E^i is the interior of E and \bar{E} is the (norm) closure of E .

Some further notations are $\text{Lat}(T)$ for the lattice of all invariant subspaces under T , and $\text{SM}(T)$ for the family of all spectral maximal spaces of T . The abbreviation svep stands for the single valued extension property.

Main results.

THEOREM 1. *Let T be S -decomposable.*

(1) *If F is a closed set with $S \subset F$, then $X_T(F) \in \text{SM}(T)$, $\overline{(F^i \setminus S) \cap \sigma(T)} \subset \sigma(T|X_T(F)) \subset F \cap \sigma(T)$ and $\overline{\sigma(T) \setminus F} \subset \sigma(T/X_T(F)) \subset \overline{\sigma(T) \setminus F} \cup S$.*

(2) *If G is an open set with $S \subset G$, then $X_T(\bar{G}) \in \text{SM}(T)$, $\overline{G \cap \sigma(T)} \subset \sigma(T|X_T(\bar{G})) \subset \bar{G} \cap \sigma(T)$ and $\sigma(T/X_T(\bar{G})) = \overline{\sigma(T) \setminus \bar{G}}$.*

(3) *If F is a closed set with $S \cap F = \emptyset$, then $X_T(F) \in \text{SM}(T)$, $\overline{F^i \cap \sigma(T)} \subset \sigma(T|X_T(F)) \subset F \cap \sigma(T)$ and $\sigma(T/X_T(F)) = \overline{\sigma(T) \setminus F}$.*

For the proof of Theorem 1 we need the following Lemma.

LEMMA. *Let T be S -decomposable. If F is a closed set with $S \subset F$ or $S \cap F = \emptyset$, then $X_T(F) \in \text{SM}(T)$, $\sigma(T|X_T(F)) \subset F$ and $X_T(F)$ is S^c -analytically invariant under T .*

PROOF. It is proved that $X_T(F) \in \text{SM}(T)$ and $\sigma(T|X_T(F)) \subset F$ by [1] (the case $S \subset F$) and [6] (the case $S \cap F = \emptyset$). Let D be an open set with $D \cap S = \emptyset$ and $f: D \rightarrow X$ be an analytic function such that $(z - T)f(z) \in X_T(F)$ on D . We may assume D is an open ball. If $D \subset \sigma(T|X_T(F))$, then $f(z) \in X_T(F)$ on D because $X_T(F)$ is T -absorbent by [3, Theorem 1.3.7]. If $D \cap \rho(T|X_T(F)) \neq \emptyset$, then $(z - T)(z - T|X_T(F))^{-1}(z - T)f(z) = (z - T)f(z)$ on $D \cap \rho(T|X_T(F))$. Since T has S^c -svep by [1], we have $f(z) = (z - T|X_T(F))^{-1}(z - T)f(z) \in X_T(F)$ on $D \cap \rho(T|X_T(F))$, hence on D .

PROOF OF THEOREM 1. (1) Since $X_T(F)$ is hyperinvariant under T , $\sigma(T|X_T(F)) \subset F \cap \sigma(T)$ by the Lemma. We prove $\sigma(T/X_T(F)) \subset \overline{\sigma(T) \setminus F} \cup S$. Let $z \notin \overline{\sigma(T) \setminus F} \cup S$. We have only to prove $z - T^F$ is surjective by the Lemma, where $T^F = T/X_T(F)$. Since there exist open sets G_1 and G_0 such that $z \in G_1$, $\bar{G}_1 \cap (\overline{\sigma(T) \setminus F} \cup S) = \emptyset$, $z \notin \bar{G}_0$ and $\{G_1; G_0\}$ is an S -covering of $\sigma(T)$, there exist $X_1, X_0 \in \text{SM}(T)$ such that $X = X_1 + X_0$ and $\sigma(T|X_i) \subset G_i$ for $i = 1, 0$. Let $x \in X$ be given. Then we can write $x = x_1 + x_0$ for some $x_i \in X_i$ for $i = 1, 0$. Then there exists $y \in X_0$ with $(z - T)y = x_0$. Since $X_1 \subset X_T(\bar{G}_1) = X_T(\bar{G}_1 \cap \sigma(T)) \subset X_T(F)$, we have $(z - T^F)\hat{y} = \hat{x}_0 = \hat{x}_1 + \hat{x}_0 = \hat{x}$ where $\hat{x} \in X/X_T(F)$ is the coset of x . Thus $z - T^F$ is surjective. Hence

$$\sigma(T|X_T(F)) \supset \sigma(T) \setminus \sigma(T/X_T(F)) \supset \sigma(T) \cap (\overline{\sigma(T) \setminus F} \cup S) \supset \sigma(T) \cap (F^i \setminus S),$$

and hence $\sigma(T|X_T(F)) \supset \overline{\sigma(T) \cap (F^i \setminus S)}$. Also

$$\sigma(T|X_T(F)) \supset \sigma(T) \setminus \sigma(T|X_T(F)) \supset \sigma(T) \setminus F,$$

and hence $\sigma(T|X_T(F)) \supset \overline{\sigma(T) \setminus F}$.

(2) We have $X_T(\overline{G}) \in \text{SM}(T)$, $\sigma(T|X_T(\overline{G})) \subset \overline{G} \cap \sigma(T)$, $\overline{\sigma(T) \setminus \overline{G}} \subset \sigma(T|X_T(\overline{G})) \subset \sigma(T) \setminus \overline{G} \cup S$ by [1]. [8, Proposition 1.4] proved $\sigma(T|X_T(\overline{G})) \subset \sigma(T) \setminus G$. Hence $\sigma(T|X_T(\overline{G})) \subset (\sigma(T) \setminus G) \cap (\sigma(T) \setminus \overline{G} \cup S) = \overline{\sigma(T) \setminus \overline{G}}$, and hence $\sigma(T|X_T(\overline{G})) = \overline{\sigma(T) \setminus \overline{G}}$. Hence $\sigma(T|X_T(\overline{G})) \supset \sigma(T) \setminus \overline{\sigma(T) \setminus \overline{G}} \supset \sigma(T) \cap G$, and hence $\sigma(T|X_T(\overline{G})) \supset \overline{\sigma(T) \cap G}$.

(3) We prove $\sigma(T|X_T(F)) \subset \overline{\sigma(T) \setminus F}$. Let $z \notin \overline{\sigma(T) \setminus F}$. We have only to prove that $z - T^F$ is surjective by the Lemma. Since there exist open sets G_1 and G_0 such that $z \in G_1$, $\overline{G_1} \cap \overline{\sigma(T) \setminus F} = \emptyset$, $z \notin \overline{G_0}$ and $\{G_1; G_0\}$ is an S -covering of $\sigma(T)$, there exist $X_1, X_0 \in \text{SM}(T)$ such that $X = X_1 + X_0$ and $\sigma(T|X_i) \subset G_i$ for $i = 1, 0$. Let $x \in X$ be given. Then we can write $x = x_1 + x_0$ for some $x_i \in X_i$ for $i = 1, 0$. Hence there exists $y \in X_0$ with $(z - T)y = x_0$. Since $X_1 \subset X_T(\overline{G_1}) = X_T(\overline{G_1} \cap \sigma(T)) \subset X_T(F)$, we have $(z - T^F)\hat{y} = \hat{x}_0 = \hat{x}_1 + \hat{x}_0 = \hat{x}$. Thus $z - T^F$ is surjective. Hence

$$\sigma(T|X_T(F)) \supset \sigma(T) \setminus \sigma(T|X_T(F)) \supset \sigma(T) \cap F^i,$$

and hence $\sigma(T|X_T(F)) \supset \overline{\sigma(T) \cap F^i}$. Also

$$\sigma(T|X_T(F)) \supset \sigma(T) \setminus \sigma(T|X_T(F)) \supset \sigma(T) \setminus F,$$

and hence $\sigma(T|X_T(F)) = \overline{\sigma(T) \setminus F}$.

REMARK. The idea of Theorem 1 is inspired by [4], which proved $\sigma(T|X_T(F)) = \overline{\sigma(T) \setminus F}$ for the strongly decomposable operator T .

THEOREM 2. Let T be S -decomposable. If G is an open set with $S \subset G$ or $S \cap \overline{G} = \emptyset$, then $X_T(G)^\perp = X_T^*(G^c)$.

PROOF. [10, Proposition 2.9] proved the case $S \cap \overline{G} = \emptyset$. We prove the case $S \subset G$. We may assume $S \subset \sigma(T)$. We prove first $X_T(G)^\perp \subset X_T^*(G^c)$. Let $x^* \in X_T(G)^\perp$. We write $G = \bigcup G_\alpha$ where G_α is a component of G . Since S is compact, we may assume that $\{\alpha | G_\alpha \cap S = \emptyset\}$ is finite. Let $S_\alpha = S \cap G_\alpha$. Then $S = \bigcup S_\alpha$ and S_α is closed. Hence there exist open sets G'_α, G''_α such that $S_\alpha \subset G'_\alpha$, $\overline{G'_\alpha} \subset G''_\alpha$ and $\overline{G''_\alpha} \subset G_\alpha$. Let $G_0 = \bigcup G''_\alpha$ and $G_1 = \bigcap (\overline{G'_\alpha})^c$. Since $\{G_1; G_0\}$ is an S -covering of $\sigma(T)$, there exist $X_1, X_0 \in \text{SM}(T)$ such that $X = X_1 + X_0$ and $\sigma(T|X_i) \subset G_i$ for $i = 1, 0$. Let $x \in X$ be given. Then $x = x_1 + x_0$ for some $x_i \in X_i$ for $i = 1, 0$. We define

$$(x_T^*(z))(z) = x^*((z - T|X_1)^{-1}x_1) \quad \text{for } z \in \bigcup G'_\alpha.$$

We prove $x_T^*(z) \in X^*$. If $x = x_1 + x_0 = x'_1 + x'_0$ for some $x'_i \in X_i$ for $i = 1, 0$, then

$$\begin{aligned} x_1 - x'_1 &= x'_0 - x_0 \in X_1 \cap X_0 \subset X_T(\overline{G_1}) \cap X_T(\overline{G_0}) \\ &= X_T(\overline{G_1} \cap \overline{G_0}) \subset X_T(G) \end{aligned}$$

because T has S^c -svcp by [1] and $S \subset G_0$. Hence $x^*((z - T|X_1)^{-1}(x_1 - x'_1)) = 0$, and hence $x_T^*(z)$ is well defined. By the Banach theorem there exists $M > 0$ such

that $\|x_1\| < M\|x\|$ for all $x \in X$. Hence

$$\begin{aligned} |(x_{T^*}^*(z))(x)| &\leq |x^*((z - T|X_1)^{-1}x_1)| \\ &\leq \|x^*\| \|(z - T|X_1)^{-1}\| M\|x\|, \end{aligned}$$

and hence $x_{T^*}^*(z) \in X^*$. Then $x_{T^*}^*(z)$ is analytic on $\bigcup G'_\alpha$ and

$$\begin{aligned} ((z - T^*)x_{T^*}^*(z))(x) &= x_{T^*}^*(z)((z - T)x) = x^*(x_1) \\ &= x^*(x_1 + x_0) = x^*(x) \end{aligned}$$

because $X_0 \subset X_T(G_0) \subset X_T(G)$. Hence $(z - T^*)x_{T^*}^*(z) = x^*$ on $\bigcup G'_\alpha$ and hence $x^* \in X_{T^*}^*((\bigcup G'_\alpha)^c)$. Since G'_α is any open set with $S_\alpha \subset G'_\alpha \subset G_\alpha$ and T has S^c -svep, we have $x^* \in X_{T^*}^*(G^c)$, and hence $X_T(G)^\perp \subset X_{T^*}^*(G^c)$. The proof of the converse inclusion is easy. Thus $X_T(G)^\perp = X_{T^*}^*(G^c)$.

By the relations $\sigma(T) = \sigma(T^*)$, $(X/Y)^* = Y^\perp$ and $X^*/Y^\perp = Y^*$ for $Y \subset X$, we have the following theorem by Theorem 1.

THEOREM 3. *Let T be S -decomposable. If G is an open set with $S \subset G$ or $S \cap \overline{G} = \emptyset$, then $\sigma(T|\overline{X_T(G)}) = \overline{G \cap \sigma(T)}$ and $\sigma(T) \setminus \overline{G \cap \sigma(T)} \subset \sigma(T|\overline{X_T(G)}) \subset \sigma(T) \setminus G$.*

It is easy to construct a decomposable operator which satisfies $\overline{F^i \cap \sigma(T)} \subseteq \sigma(T|\overline{X_T(F)}) \subseteq F \cap \sigma(T)$ for some closed set $F \subset \overline{C}$. We consider $X_T(\overline{G \cap \sigma(T)})$ where $G \subset \overline{C}$ is open. It is clear that

$$\overline{X_T(G)} = \overline{X_T(G \cap \sigma(T))} \subset X_T(\overline{G \cap \sigma(T)}) \subset X_T(\overline{G}).$$

We prove that $X_T(\overline{G \cap \sigma(T)})$ has an interesting property.

THEOREM 4. *Let T be S -decomposable. If G is an open set with $S \subset G$ or $S \cap \overline{G} = \emptyset$, then $X_T(\overline{G \cap \sigma(T)}) \in \text{SM}(T)$,*

$$\begin{aligned} \sigma(T|\overline{X_T(G \cap \sigma(T))}) &= \overline{G \cap \sigma(T)}, \\ \sigma(T/X_T(\overline{G \cap \sigma(T)})) &= \overline{\sigma(T) \setminus \overline{G \cap \sigma(T)}}. \end{aligned}$$

PROOF. We prove first the case $S \cap \overline{G} = \emptyset$. By Theorem 1, we have only to prove $\overline{G \cap \sigma(T)} \subset \sigma(T|\overline{X_T(G \cap \sigma(T))})$. Let $z \in G \cap \sigma(T)$ be given. Then there exist open sets G_1, G_0 such that $z \in G_1, \overline{G_1} \subset G, z \notin \overline{G_0}$ and $\{G_1; G_0\}$ is an S -covering of $\sigma(T)$. Hence there exist open sets G_1, G_0 such that $z \in G_1, \overline{G_1} \subset G, z \notin G_0$ and $\{G_1; G_0\}$ is an S -covering of $\sigma(T)$. Hence there exist $X_1, X_0 \in \text{SM}(T)$ such that $X = X_1 + X_0$ and $\sigma(T|X_i) \subset G_i$ for $i = 1, 0$. Then

$$X_1 \subset X_T(\overline{G_1}) \subset X_T(G) = X_T(G \cap \sigma(T)) \subset X_T(\overline{G \cap \sigma(T)}),$$

and $X_0 \subset X_T(\overline{G_0})$. Hence

$$X = X_T(\overline{G \cap \sigma(T)}) + X_T(\overline{G_0}).$$

Then it is easy to prove

$$\sigma(T) \subset (T|\overline{X_T(G \cap \sigma(T))}) \cup \sigma(T|\overline{X_T(G_0)}) \cup S.$$

Since $z \in \sigma(T), z \notin \sigma(T|\overline{X_T(G_0)})$ and $z \notin S$, we have $z \in \sigma(T|\overline{X_T(G \cap \sigma(T))})$. Hence $G \cap \sigma(T) \subset \sigma(T|\overline{X_T(G \cap \sigma(T))})$, and $\overline{G \cap \sigma(T)} \subset \sigma(T|\overline{X_T(G \cap \sigma(T))})$.

We prove the case $S \subset G$. By Theorem 1(1), $X_T(\overline{G \cap \sigma(T)}) \in \text{SM}(T)$ and $\sigma(T|X_T(\overline{G \cap \sigma(T)})) \subset \overline{G \cap \sigma(T)}$. We prove $\sigma(T/X_T(\overline{G \cap \sigma(T)})) \subset G^c$. Let $z \in G$ be given. Then there exist open sets G_1, G_0 such that $z \in G_0, S \subset G_0, \overline{G_0} \subset G, z \notin \overline{G_1}$ and $\{G_1; G_0\}$ is an S -covering of $\sigma(T)$. Hence there exist $X_1, X_0 \in \text{SM}(T)$ such that $X = X_1 + X_0$ and $\sigma(T|X_i) \subset G_i$ for $i = 1, 0$. Since

$$X_0 \subset X_T(\overline{G_0}) \subset X_T(G) = X_T(G \cap \sigma(T)) \subset X_T(\overline{\sigma(T) \cap G}),$$

we have $X = X_T(\overline{G \cap \sigma(T)}) + X_T(\overline{G_1})$. Since T has S^c -svep and $X_T(\overline{G_1})$ is closed, we have

$$\begin{aligned} X/X_T(\overline{G \cap \sigma(T)}) &= X_T(\overline{G_1})/(X_T(\overline{G \cap \sigma(T)}) \cap X_T(\overline{G_1})) \\ &= X_T(\overline{G_1})/X_T(\overline{G \cap \sigma(T)} \cap \overline{G_1}). \end{aligned}$$

Then

$$\sigma(T/X_T(\overline{G \cap \sigma(T)})) \subset \sigma(T|X_T(\overline{G_1})) \cup \sigma(T|X_T(\overline{G \cap \sigma(T)})) \subset \overline{G_1},$$

and hence $z \in \rho(T/X_T(\overline{G \cap \sigma(T)}))$. Hence, by Theorem 1(1),

$$\sigma(T/X_T(\overline{G \cap \sigma(T)})) \subset G^c \cap \overline{\{\sigma(T) \setminus \overline{G \cap \sigma(T)} \cup S\}} = \overline{\sigma(T) \setminus \overline{G \cap \sigma(T)}},$$

and hence $\sigma(T/X_T(\overline{G \cap \sigma(T)})) = \overline{\sigma(T) \setminus \overline{G \cap \sigma(T)}}$. Hence

$$\sigma(T|X_T(\overline{G \cap \sigma(T)})) \supset \sigma(T) \setminus \sigma(T/X_T(\overline{G \cap \sigma(T)})) \supset \overline{G \cap \sigma(T)},$$

and hence $\sigma(T|X_T(\overline{G \cap \sigma(T)})) = \overline{G \cap \sigma(T)}$.

REMARK. Hence $\overline{G \cap \sigma(T)}$ is a set spectrum of T (cf. [3]). Theorems 1, 3 and 4 imply that if T is S -decomposable, then the mappings $G \rightarrow X_T(\overline{G}), \overline{X_T(\overline{G})}, X_T(\overline{G \cap \sigma(T)})$ are, we may say, “ S -spectral resolvents” (cf. [2]). The following theorem proves that the converse implication is also true.

THEOREM 5. (1) *If, for every open set with $S \subset G$, there exists $Y \in \text{Lat}(T)$ such that $\sigma(T|Y) \subset \overline{G}$ and $\sigma(T/Y) \subset G^c \cup S$, then T is S -decomposable.*

(2) *If, for every open set G with $S \cap \overline{G} = \emptyset$, there exists $Y \in \text{Lat}(T)$ such that $\sigma(T|Y) \subset \overline{G} \cup S$ and $\sigma(T/Y) \subset G^c$, then T is S -decomposable.*

PROOF. [6] proved (1). We prove (2) by using (1). Let G be an open set with $S \subset G$. Then $G_1 = \overline{G}^c$ is open and $\overline{G_1} \cap S = \emptyset$. Hence there exists $Y \in \text{Lat}(T)$ such that $\sigma(T|Y) \subset \overline{G_1} \cup S$ and $\sigma(T/Y) \subset G_1^c$. Then

$$\sigma(T^*/Y^\perp) = \sigma((T|Y)^*) = \sigma(T|Y) \subset G_1 \cup S \subset G^c \cup S,$$

and

$$\sigma(T^*|Y^\perp) = \sigma((T/Y)^*) = \sigma(T/Y) = \overline{G_1}^c = \overline{G}.$$

Hence T^* is S -decomposable by (1). Hence T is S -decomposable by [9].

REMARK. [9] proved (2) under the conditions that $\sigma(T|Y) \subset \overline{G}$ and $\sigma(T/Y) \subset G^c \cap \sigma(T)$.

THEOREM 6. *Let T be S -decomposable. If F is a closed set with $S \subset F$ or $S \cap F = \emptyset$, then $JX_T(F) = JX \cap X_T^{**}(F)$ where $J: X \rightarrow X^{**}$ is the canonical embedding.*

PROOF. [9] proved the case $S \subset F$. We prove the case $S \cap F = \emptyset$. Let $Jx \in X_T^{**}(F)$ be given. Then there exists an analytic function $f: F^c \rightarrow X^{**}$ such

that $(z - T^{**})f(z) = Jx$ on F^c . We write $F^c = \bigcup G_\alpha$ where G_α is a component of F^c . Since JX is S^c -analytically invariant under T^{**} by [9, Theorem 2.2] and $G_\alpha \cap S^c = \emptyset$, we have $f(z) \in JX$ on $G_\alpha \cap S^c$, hence on G_α . This implies $J^{-1}f(z) \in X$ exist for all $z \in \bigcup G_\alpha = F^c$. Hence $(z - T)J^{-1}f(z) = x$ on F^c , and hence $x \in X_T(F)$. Thus $JX \cap X_{T^{**}}(F) \subset JX_T(F)$. The proof of the converse inclusion is easy, hence the proof is finished.

THEOREM 7. *Let T be S -decomposable. If F is a closed set with $S \subset F$ or $S \cap F = \emptyset$, then $X_T(F)^\perp =$ the weak $*$ -closure of $X_{T^*}(F^c)$.*

PROOF. We remark T^* is S -decomposable by [10]. Let $f \in X_T(F)^\perp$ and $f \notin$ the weak $*$ -closure of $X_{T^*}(F^c)$. Then there exists $x \in X$ such that $x \perp X_{T^*}(F^c)$ and $f(x) = 1$. Hence $Jx \in X_{T^*}(F^c)^\perp = X_{T^{**}}(F)$ by Theorem 2, and $Jx \in JX \cap X_{T^{**}}(F) = JX_T(F)$. Hence $x \in X_T(F)$. This is a contradiction. Hence $X_T(F)^\perp \subset$ the weak $*$ -closure of $X_{T^*}(F^c)$. The proof of the converse inclusion is easy, hence the proof is finished.

REMARK. If T is S -decomposable and F is a closed set with $S \subset F$ or $S \cap F = \emptyset$, then ${}^\perp X_{T^*}(F) = \{x \in X \mid x \perp X_{T^*}(F)\} = {}^\perp (X_T(F^c)^\perp) = \overline{X_T(F^c)}$ and $X_{T^*}(F)^\perp = X_T(F^c)^\perp =$ the weak $*$ -closure of $X_T(F^c)$ in X^{**} .

ACKNOWLEDGMENT. The author wishes to thank Professor T. Yoshino for his helpful comments.

REFERENCES

1. I. Bacalu, *S-decomposable operators in Banach space*, Rev. Roumaine Math. Pures Appl. **20** (1975), 1101–1107.
2. I. Erdelyi, *Spectral resolvents*, Operator Theory and Functional Analysis, Research Notes in Math., no. 38, Pitman, San Francisco, Calif., 1979, pp. 51–70.
3. I. Erdelyi and R. Lange, *Spectral decompositions of Banach spaces*, Springer-Verlag, New York, 1977.
4. I. Erdelyi and W. Shengwang, *On strongly decomposable operators*, Pacific J. Math. **110** (1984), 287–296.
5. J. K. Finch, *The single valued extension property on a Banach space*, Pacific J. Math. **58** (1975), 61–69.
6. K. Tanahashi, *Characterizations of S-decomposable operators on a complex Banach space*, Tôhoku Math. J. **35** (1983), 261–265.
7. B. Nagy, *On S-decomposable operators*, J. Operator Theory **2** (1979), 277–286.
8. M. Radjabalipour, *On subnormal operators*, Trans. Amer. Math. Soc. **211** (1975), 377–389.
9. W. Shengwang and L. Guangyu, *On the duality theorem of bounded S-decomposable operators*, J. Math. Anal. Appl. **99** (1984), 150–163.
10. F.-H. Vasilescu, *On the residual decomposability in dual spaces*, Rev. Roumaine Math. Pures Appl. **16** (1971), 1573–1587.

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