NOTES ON INTERPOLATION BY THE REAL METHOD BETWEEN \(C(T, A_0)\) AND \(C(T, A_1)\)

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ABSTRACT. Let \(A\) be a Banach space and let \(T\) be a compact Hausdorff space. We denote by \(C(T, A)\) the Banach space of all \(A\)-valued continuous functions defined on \(T\) endowed with the supremum norm. We show that if \(T\) is infinite and \((A_0, A_1)\) is a Banach couple with \(A_0\) continuously embedded in \(A_1\), then the interpolation space \((C(T, A_0), C(T, A_1))_{\varphi, p}\) is equal to \(C(T, (A_0, A_1)_{\varphi, p})\) if and only if \(A_0\) is closed in \(A_1\).

A pair \(\mathcal{A} = (A_0, A_1)\) of Banach spaces is called a Banach couple if \(A_0\) and \(A_1\) are both continuously embedded in some Hausdorff topological vector space \(V\). The Peetre \(K\)-functional of an element \(a\) in \(A_0 + A_1\) is defined for \(s > 0\) by

\[
K(s, a; \mathcal{A}) = \inf \{ \|a_0\|_{A_0} + s\|a_1\|_{A_1} : a = a_0 + a_1, \ a_0 \in A_0, \ a_1 \in A_1 \}.
\]

Let \(\mathcal{B}\) denote the set of all positive continuous functions on \((0, \infty)\) such that \(0 < \alpha_{\varphi} \leq \beta_{\varphi} < 1\), where \(\varphi(s) = \sup_{u>0}(\varphi(us)/\varphi(u))\) and

\[
\alpha_{\varphi} = \sup_{0 < s < 1} \frac{\ln \varphi(s)}{\ln s}, \quad \beta_{\varphi} = \inf_{1 < s < \infty} \frac{\ln \varphi(s)}{\ln s}.
\]

Let \(\mathcal{A} = (A_0, A_1)\) be a Banach couple, \(1 \leq p < \infty\) and \(\varphi \in \mathcal{B}\). The interpolation Banach space \(\mathcal{A}_{\varphi, p} = (A_0, A_1)_{\varphi, p}\) consists of all \(a \in A_0 + A_1\) for which

\[
\|a\|_{\varphi, p} = \left( \int_0^\infty (\varphi(s)^{-1} K(s, a; \mathcal{A}))^p ds/s \right)^{1/p} < \infty.
\]

If \(\varphi(s) = s^\theta\) (0 < \(\theta\) < 1) we write, shortly, \(\mathcal{A}_{\theta, p}\) (see [1, 5, 6] for details). We note that \(A_0 \cap A_1\) is a dense subspace of \(\mathcal{A}_{\varphi, p}\).

The purpose of this paper is to investigate the connection between the spaces \((C(T, A_0), C(T, A_1))_{\varphi, p}\) and \(C(T, (A_0, A_1)_{\varphi, p})\) for \(A_0 \hookrightarrow A_1\) (the symbol \(\hookrightarrow\) denotes continuous inclusion), where \(T\) is a compact Hausdorff space. In Bona and Scott [2], an interesting application of the following continuous inclusion is shown:

\[
(*) \quad (C(T, A_0), C(T, A_1))_{\varphi, p} \hookrightarrow C(T, (A_0, A_1)_{\varphi, p}),
\]

where \(\varphi(s) = s^\theta, \ T = [0, a], \ a > 0,\) in the theory of the Korteweg-de Vries equation. We will show that if \(T\) is an infinite compact Hausdorff space, then equality in (*) holds if and only if the Banach couple \((A_0, A_1)\) is trivial, i.e. \(A_0\) is a closed subspace in \(A_1\). In order to show this fact, we shall first establish some results.
PROPOSITION 1. Let \((A_0, A_1)\) be a Banach couple such that \(A_0 \hookrightarrow A_1\). If \(1 \leq p, q < \infty\) and \(\varphi, \psi \in \mathcal{B}\) with \(\beta < \alpha, \psi\), then \((A_0, A_1)_{\varphi, p} \hookrightarrow (A_0, A_1)_{\psi, q}\).

An easy proof of this proposition can be found in [4].

PROPOSITION 2. Let \(\tilde{A} = (A_0, A_1)\) be a nontrivial Banach couple such that \(A_0 \hookrightarrow A_1, \varphi \in \mathcal{B}, 1 \leq p < \infty\), and let \(E\) be an infinite-dimensional closed subspace of \(\tilde{A}_{\varphi, p}\). If \(E\) is not closed in \(A_1\), then \(E\) contains a closed complemented in \(\tilde{A}_{\varphi, p}\) subspace \(F\) isomorphic to \(l^p\).

PROOF. Take \(\theta \in (0, 1)\) such that \(\beta < \theta\) and \(\eta \in (0, 1)\) with \(\eta < \alpha\). If we put \(\varphi_0(s) = \varphi(s)\gamma s^\theta\) for \(s > 0\), where \(\gamma = 1/1 - \eta\) and \(\beta = \eta \gamma /1 - \eta\), we obtain \(\alpha = -\beta + \gamma \alpha\), \(\beta_0 = -\beta + \gamma \beta\). Hence

\[
\begin{align*}
0 &< \alpha < \beta < \theta < 1,
\end{align*}
\]

so \(\varphi_0\) belongs to \(\mathcal{B}\). Since \(\varphi_0(s) (s^\theta / \varphi_0(s)) = \varphi(s)\), we see that

\[
\begin{align*}
(\tilde{A}_{\varphi_0, p}, \tilde{A}_{\theta, p}) &\hookrightarrow \tilde{A}_{\varphi_0, p}
\end{align*}
\]

by Theorem 5.4.1 in [1] and reiteration (see [3, 10]).

From Proposition 1, we have

\[
\begin{align*}
X_0 = \tilde{A}_{\varphi_0, p} &\hookrightarrow \tilde{A}_{\theta, p} = X_1
\end{align*}
\]

by (1). We observe that \(X_0\) is not closed in \(X_1\). To the contrary, if \(X_0\) is closed in \(X_1\), then \(X_0 = X_1\) by density of \(X_0\) in \(X_1\). So, if we take \(\theta_1\) such that \(\beta_0 < \theta_1 < \theta\), then \(\tilde{A}_{\theta, p} = \tilde{A}_{\theta_1, p}\) by (3) and Proposition 1, a contradiction with Theorem 1 in [7].

Now let \(E\) be an infinite-dimensional subspace of \(\tilde{A}_{\varphi, p}\) such that \(E\) is not closed in \(A_1\). We get

\[
\begin{align*}
E &\subset \tilde{A}_{\varphi, p} \hookrightarrow \tilde{A}_{\theta, p}
\end{align*}
\]

by (2). We have two mutually exclusive possibilities for \(E\):

1°. \(E\) is closed in \(\tilde{A}_{\theta, p}\). Then \(E\) contains a closed complemented in \(\tilde{A}_{\theta, p}\) subspace \(F\) isomorphic to \(l^p\) by Theorem 1 in [8]. Obviously, \(F\) is a closed complemented in \(\tilde{A}_{\varphi, p}\) subspace and isomorphic to \(l^p\) by the closed graph theorem and continuous inclusion (4).

2°. \(E\) is not closed in \(\tilde{A}_{\theta, p}\). Then \(E\) contains a closed complemented in \((X_0, X_1)_{\eta, p}\) subspace isomorphic to \(l^p\) by results of [8]. So by (2) we obtain our assertion. The proof is complete.

THEOREM 1. Let \((A_0, A_1)\) be a nontrivial Banach couple with \(A_0 \hookrightarrow A_1, \varphi \in \mathcal{B}, 1 \leq p < \infty\). If \(T\) is an infinite compact Hausdorff space, then

\[
(C(T, A_0), C(T, A_1))_{\varphi, p} \neq C(T, A)
\]

for each Banach space \(A\) intermediate between \(A_0\) and \(A_1\) (i.e. \(A_0 \hookrightarrow A \hookrightarrow A_1\), which is not closed in \(A_1\).

PROOF. Assume that a Banach space \(A\) intermediate between \(A_0\) and \(A_1\) is not closed in \(A_1\), and

\[
(C(T, A_0), C(T, A_1))_{\varphi, p} = C(T, A)
\]
Since $A$ is not closed in $A_1$, there exists a sequence $(a_n)$ in $A$ such that $\|a_n\|_{A_1} \rightarrow 0$. On the other hand, since $T$ is infinite, there exists a sequence $(U_n)$ of open nonempty pairwise disjoint subsets of $T$. For each $n \in \mathbb{N}$ choose $t_n \in U_n$. An application of Urysohn’s lemma shows that for each $n \in \mathbb{N}$ there exist functions $f_n \in C(K)$ such that

$$f_n(t_n) = 1, \quad f_n(t) = 0 \quad \text{for } t \in T \setminus U_n \quad \text{and} \quad \|f_n\|_{C(K)} = 1.$$  

Let $(f_n \otimes a_n)(t) := f_n(t)a_n$ for $n \in \mathbb{N}$ and $t \in T$. If $(\xi_i) \in \mathbb{R}^N$, then for $m > n > 1$ we have

$$\left\| \sum_{i=n}^m \xi_i f_i \otimes a_i \right\|_{C(K,A)} = \sup_{t \in T} \left\| \sum_{i=n}^m \xi_i f_i(t)a_i \right\|_A = \sup \{|\xi_i|: n \leq i \leq m\},$$

by (6). Hence $\sum_{n=1}^{\infty} \xi_n f_n \otimes a_n$ is a norm convergent series in $C(T, A)$ if and only if $\xi = (\xi_n) \in c_0$, and

$$\left\| \sum_{n=1}^{\infty} \xi_n f_n \otimes a_n \right\|_{C(T,A)} = \|\xi\|_{c_0}.$$ 

This shows that the space $E = \text{span}\{f_n \otimes a_n: n \in \mathbb{N}\}$, the closed linear span in $C(T, A)$ of $\{f_n \otimes a_n: n \in \mathbb{N}\}$, is isometrically isomorphic to $c_0$.

Let us now note that $E$ is not closed in $C(T, A_1)$. So, if we suppose that $E$ is closed in $C(T, A_1)$, then there exists a constant $c > 0$ such that $\|f\|_{C(T, A_1)} \geq c\|f\|_{C(T, A)}$ for each $f \in E$ (by the closed graph theorem and continuous inclusion $C(T, A) \hookrightarrow C(T, A_1)$). Hence for $f = f_n \otimes a_n \in E$, $n \in \mathbb{N}$, we have $\|f\|_{C(T, A_1)} = \|a_n\|_{A_1} \geq c\|a_n\|_A = c$ by (6), a contradiction, because $a_n \rightarrow 0$ in $A_1$.

Since the couple $(A_0, A_1)$ is nontrivial, it follows that $C(T, A_0)$ is not closed in $C(T, A_1)$; Proposition 2 and equality (5) give that $E$ contains a subspace closed in $C(T, A)$ isomorphic to $l^p$. This implies that $l^p$ is isomorphic to a subspace of $c_0$. This contradiction finishes the proof.

**Corollary 1.** (a) Let $(A_0, A_1)$ be a Banach couple, $\varphi \in \mathcal{B}$, $1 \leq p < \infty$, and let $T$ be a compact Hausdorff space. Then

$$\left(C(T, A_0), C(T, A_1)\right)^{\varphi, p} \hookrightarrow C(T, (A_0, A_1)^{\varphi, p}).$$

(b) If $A_0 \hookrightarrow A_1$ and $T$ is infinite, then equality in (7) holds if and only if $A_0$ is closed in $A_1$.

**Proof.** (a) It is easy to see that

$$K(s, f; C(T, A_0), C(T, A_1)) \geq \sup_{t \in T} K(s, f(t); A_0, A_1)$$

for $f$ in $C(T, A_0) + C(T, A_1)$ and $s > 0$, and the proof of (7) is analogous to the proof of Proposition 3 in [2].

(b) Let $A_0 \hookrightarrow A_1$. If $A_0$ is closed in $A_1$, then $(A_0, A_1)^{\varphi, p} = A_0$ by density of $A_0$ in $(A_0, A_1)^{\varphi, p}$. Hence we obtain that

$$(C(T, A_0), C(T, A_1))^{\varphi, p} = C(T, (A_0, A_1)^{\varphi, p}) = C(T, A_0).$$

Now let $T$ be infinite and assume that $A_0$ is not closed in $A_1$. Then $(A_0, A_1)^{\varphi, p}$ is not closed in $A_1$. Assuming otherwise, $(A_0, A_1)^{\varphi, p} = (A_0^0, A_1^0)^{\varphi, p} = A_1^0$, where $A_0^0$ is the closure of $A_0$ in $A_1$. But this contradicts [9, Lemma 2], by $\varphi \in \mathcal{B}$. So Theorem 1 applies, and the proof is complete.
References


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