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NOTES ON INTERPOLATION BY THE REAL METHOD BETWEEN $C(T, A_0)$ AND $C(T, A_1)$

MIECZYSLAW MASTYLO

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ABSTRACT. Let A be a Banach space and let T be a compact Hausdorff space. We denote by $C(T, A)$ the Banach space of all A-valued continuous functions defined on T endowed with the supremum norm. We show that if T is infinite and (A_0, A_1) is a Banach couple with A_0 continuously embedded in A_1, then the interpolation space $(C(T, A_0), C(T, A_1))_{\varphi,p}$ is equal to $C(T, (A_0, A_1)_{\varphi,p})$ if and only if A_0 is closed in A_1.

A pair $\vec{A} = (A_0, A_1)$ of Banach spaces is called a *Banach couple* if A_0 and A_1 are both continuously embedded in some Hausdorff topological vector space V. The Peetre *K-functional* of an element a in $A_0 + A_1$ is defined for $s > 0$ by

$$K(s, a; \vec{A}) = \inf\{\|a_0\|_{A_0} + s\|a_1\|_{A_1} : a = a_0 + a_1,\ a_0 \in A_0,\ a_1 \in A_1\}.$$

Let $\mathcal{B}$ denote the set of all positive continuous functions on $(0, \infty)$ such that $0 < \alpha_{\bar{\varphi}} \leq \beta_{\bar{\varphi}} < 1$, where $\bar{\varphi}(s) = \sup_{u>0}(\varphi(us)/\varphi(u))$ and

$$\alpha_{\bar{\varphi}} = \sup_{0<s<1} \frac{\ln \bar{\varphi}(s)}{\ln s}, \qquad \beta_{\bar{\varphi}} = \inf_{1<s<\infty} \frac{\ln \bar{\varphi}(s)}{\ln s}.$$

Let $\vec{A} = (A_0, A_1)$ be a Banach couple, $1 \leq p < \infty$ and $\varphi \in \mathcal{B}$. The *interpolation Banach space* $\vec{A}_{\varphi,p} = (A_0, A_1)_{\varphi,p}$ consists of all $a \in A_0 + A_1$ for which

$$\|a\|_{\varphi,p} = \left(\int_0^\infty (\varphi(s)^{-1} K(s, a; \vec{A}))^p \, ds/s\right)^{1/p} < \infty.$$

If $\varphi(s) = s^\theta$ $(0 < \theta < 1)$ we write, shortly, $\vec{A}_{\theta,p}$ (see [**1, 5, 6**] for details). We note that $A_0 \cap A_1$ is a dense subspace of $\vec{A}_{\varphi,p}$.

The purpose of this paper is to investigate the connection between the spaces $(C(T, A_0), C(T, A_1))_{\varphi,p}$ and $C(T, (A_0, A_1)_{\varphi,p})$ for $A_0 \hookrightarrow A_1$ (the symbol $\hookrightarrow$ denotes continuous inclusion), where T is a compact Hausdorff space. In Bona and Scott [**2**], an interesting application of the following continuous inclusion is shown:

$$(*) \qquad (C(T, A_0), C(T, A_1))_{\varphi,p} \hookrightarrow C(T, (A_0, A_1)_{\varphi,p}),$$

where $\varphi(s) = s^\theta$, $T = [0, a]$, $a > 0$, in the theory of the Korteweg-de Vries equation. We will show that if T is an infinite compact Hausdorff space, then equality in $(*)$ holds if and only if the Banach couple (A_0, A_1) is *trivial*, i.e. A_0 is a closed subspace in A_1. In order to show this fact, we shall first establish some results.

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PROPOSITION 1. *Let* (A_0, A_1) *be a Banach couple such that* $A_0 \hookrightarrow A_1$. *If* $1 \le p, q < \infty$ *and* $\varphi, \psi \in \mathcal{B}$ *with* $\beta_{\bar\varphi} < \alpha_{\bar\psi}$, *then* $(A_0, A_1)_{\varphi,p} \hookrightarrow (A_0, A_1)_{\psi,q}$.

An easy proof of this proposition can be found in [**4**].

PROPOSITION 2. *Let* $\vec{A} = (A_0, A_1)$ *be a nontrivial Banach couple such that* $A_0 \hookrightarrow A_1$, $\varphi \in \mathcal{B}$, $1 \le p < \infty$, *and let* E *be an infinite-dimensional closed subspace of* $\vec{A}_{\varphi,p}$. *If* E *is not closed in* A_1, *then* E *contains a closed complemented in* $\vec{A}_{\varphi,p}$ *subspace* F *isomorphic to* l^p.

PROOF. Take $\theta \in (0,1)$ such that $\beta_{\bar\varphi} < \theta$ and $\eta \in (0,1)$ with $\eta\theta < \alpha_{\bar\varphi}$. If we put $\varphi_0(s) = \varphi(s)^\gamma / s^\beta$ for $s > 0$, where $\gamma = 1/1-\eta$ and $\beta = \theta\eta/1-\eta$, we obtain $\alpha_{\bar\varphi_0} = -\beta + \gamma\alpha_{\bar\varphi}$, $\beta_{\bar\varphi_0} = -\beta + \gamma\beta_{\bar\varphi}$. Hence

$$0 < \alpha_{\bar\varphi_0} \le \beta_{\bar\varphi_0} < \theta < 1, \tag{1}$$

so φ_0 belongs to $\mathcal{B}$. Since $\varphi_0(s)(s^\theta/\varphi_0(s))^\eta = \varphi(s)$, we see that

$$(\vec{A}_{\varphi_0,p}, \vec{A}_{\theta,p})_{\eta,p} = \vec{A}_{\varphi,p} \tag{2}$$

by Theorem 5.4.1 in [**1**] and reiteration (see [**3**, **10**]).

From Proposition 1, we have

$$X_0 = \vec{A}_{\varphi_0,p} \hookrightarrow \vec{A}_{\theta,p} = X_1 \tag{3}$$

by (1). We observe that X_0 is not closed in X_1. To the contrary, if X_0 is closed in X_1, then $X_0 = X_1$ by density of X_0 in X_1. So, if we take θ_1 such that $\beta_{\bar\varphi_0} < \theta_1 < \theta$, then $\vec{A}_{\theta,p} = \vec{A}_{\theta_1,p}$ by (3) and Proposition 1, a contradiction with Theorem 1 in [**7**].

Now let E be an infinite-dimensional subspace of $\vec{A}_{\varphi,p}$ such that E is not closed in A_1. We get

$$E \subset \vec{A}_{\varphi,p} \hookrightarrow \vec{A}_{\theta,p} \tag{4}$$

by (2). We have two mutually exclusive possibilities for E:

1°. E is closed in $\vec{A}_{\theta,p}$. Then E contains a closed complemented in $\vec{A}_{\theta,p}$ subspace F isomorphic to l^p by Theorem 1 in [**8**]. Obviously, F is a closed complemented in $\vec{A}_{\varphi,p}$ subspace and isomorphic to l^p by the closed graph theorem and continuous inclusion (4).

2°. E is not closed in $\vec{A}_{\theta,p}$. Then E contains a closed complemented in $(X_0, X_1)_{\eta,p}$ subspace isomorphic to l^p by results of [**8**]. So by (2) we obtain our assertion. The proof is complete.

THEOREM 1. *Let* (A_0, A_1) *be a nontrivial Banach couple with* $A_0 \hookrightarrow A_1$, $\varphi \in \mathcal{B}$, $1 \le p < \infty$. *If* T *is an infinite compact Hausdorff space, then*

$$(C(T, A_0), C(T, A_1))_{\varphi,p} \ne C(T, A)$$

for each Banach space A *intermediate between* A_0 *and* A_1 (*i.e.* $A_0 \hookrightarrow A \hookrightarrow A_1$), *which is not closed in* A_1.

PROOF. Assume that a Banach space A intermediate between A_0 and A_1 is not closed in A_1, and

$$(C(T, A_0), C(T, A_1))_{\varphi,p} = C(T, A). \tag{5}$$

Since A is not closed in A_1, there exists a sequence (a_n) in A such that $\|a_n\|_A = 1$ and $\|a_n\|_{A_1} \to 0$. On the other hand, since T is infinite, there exists a sequence (U_n) of open nonempty pairwise disjoint subsets of T. For each $n \in \mathbf{N}$ choose $t_n \in U_n$. An application of Urysohn's lemma shows that for each $n \in \mathbf{N}$ there exist functions $f_n \in C(K)$ such that

$$(6) \qquad f_n(t_n) = 1, \quad f_n(t) = 0 \quad \text{for } t \in T \setminus U_n \quad \text{and} \quad \|f_n\|_{C(K)} = 1.$$

Let $(f_n \otimes a_n)(t) := f_n(t)a_n$ for $n \in \mathbf{N}$ and $t \in T$. If $(\xi_i) \in \mathbf{R}^{\mathbf{N}}$, then for $m > n \geq 1$ we have

$$\left\|\sum_{i=n}^{m} \xi_i f_i \otimes a_i\right\|_{C(K,A)} = \sup_{t\in T}\left\|\sum_{i=n}^{m} \xi_i f_i(t) a_i\right\|_A = \sup\{|\xi_i| \colon n \leq i \leq m\},$$

by (6). Hence $\sum_{n=1}^{\infty} \xi_n f_n \otimes a_n$ is a norm convergent series in $C(T,A)$ if and only if $\xi = (\xi_n) \in c_0$, and

$$\left\|\sum_{n=1}^{\infty} \xi_n f_n \otimes a_n\right\|_{C(T,A)} = \|\xi\|_{c_0}.$$

This shows that the space $E = \overline{\operatorname{span}}\{f_n \otimes a_n \colon n \in \mathbf{N}\}$, the closed linear span in $C(T,A)$ of $\{f_n \otimes a_n \colon n \in \mathbf{N}\}$, is isometrically isomorphic to c_0.

Let us now note that E is not closed in $C(T,A_1)$. So, if we suppose that E is closed in $C(T,A_1)$, then there exists a constant $c > 0$ such that $\|f\|_{C(T,A_1)} \geq c\|f\|_{C(T,A)}$ for each $f \in E$ (by the closed graph theorem and continuous inclusion $C(T,A) \hookrightarrow C(T,A_1)$). Hence for $f = f_n \otimes a_n \in E$, $n \in \mathbf{N}$, we have $\|f\|_{C(T,A_1)} = \|a_n\|_{A_1} \geq c\|a_n\|_A = c$ by (6), a contradiction, because $a_n \to 0$ in A_1.

Since the couple (A_0, A_1) is nontrivial, it follows that $C(T,A_0)$ is not closed in $C(T,A_1)$; Proposition 2 and equality (5) give that E contains a subspace closed in $C(T,A)$ isomorphic to l^p. This implies that l^p is isomorphic to a subspace of c_0. This contradiction finishes the proof.

COROLLARY 1. (a) *Let (A_0, A_1) be a Banach couple, $\varphi \in \mathcal{B}$, $1 \leq p < \infty$, and let T be a compact Hausdorff space. Then*

$$(7) \qquad (C(T,A_0), C(T,A_1))_{\varphi,p} \hookrightarrow C(T,(A_0,A_1)_{\varphi,p}).$$

(b) *If $A_0 \hookrightarrow A_1$ and T is infinite, then equality in* (7) *holds if and only if A_0 is closed in A_1.*

PROOF. (a) It is easy to see that

$$K(s,f;C(T,A_0),C(T,A_1)) \geq \sup_{t\in T} K(s,f(t);A_0,A_1)$$

for f in $C(T,A_0) + C(T,A_1)$ and $s > 0$, and the proof of (7) is analogous to the proof of Proposition 3 in [**2**].

(b) Let $A_0 \hookrightarrow A_1$. If A_0 is closed in A_1, then $(A_0,A_1)_{\varphi,p} = A_0$ by density of A_0 in $(A_0,A_1)_{\varphi,p}$. Hence we obtain that

$$(C(T,A_0), C(T,A_1))_{\varphi,p} = C(T,(A_0,A_1)_{\varphi,p}) = C(T,A_0).$$

Now let T be infinite and assume that A_0 is not closed in A_1. Then $(A_0,A_1)_{\varphi,p}$ is not closed in A_1. Assuming otherwise, $(A_0,A_1)_{\varphi,p} = (A_0,A_1^0)_{\varphi,p} = A_1^0$, where A_1^0 is the closure of A_0 in A_1. But this contradicts [**9**, Lemma 2], by $\varphi \in \mathcal{B}$. So Theorem 1 applies, and the proof is complete.

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INSTITUTE OF MATHEMATICS, A. MICKIEWICZ UNIVERSITY, UL. MATEJKI 48/49, 60-769 POZNAŃ, POLAND