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PURE STATE EXTENSIONS OF THE TRACE ON THE CHOI ALGEBRA

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ABSTRACT. Pure state extensions of a nonhyperfinite II_1-factor state on the Choi algebra to the Cuntz algebra O_2 is constructed.

1. Introduction. Let $\mathbf{A}$ be a C^*-algebra. A state φ of $\mathbf{A}$ is a *factor state* if the closure of the image of $\mathbf{A}$ under the GNS representation induced by φ in the weak operator topology is a factor. We call this the image factor induced by φ. We will say that φ is a factor state of type I, II_1, II_∞, or III if the image factor induced by φ is, respectively, type I, II_1, II_∞, or III. If $\mathbf{B}$ is a C^*-subalgebra of $\mathbf{A}$, an old question in the theory of operator algebras asks if every factor state of $\mathbf{B}$ extends to a factor state of $\mathbf{A}$. This was answered in the affirmative recently for a separable C^*-subalgebra $\mathbf{B}$ in independent work of Longo [**6**] and Popa [**7**]. They each completed a procedure first outlined by Sakai [**8**] to obtain the desired extensions.

Given the validity of this extension procedure, a natural problem to consider next is the relation, if any, between the type of the image factor induced by the state φ of $\mathbf{B}$ and the type of the image factor induced by an extension of φ to a factor state $\mathbf{A}$. In a talk given at the conference on Operator Algebras and K-theory at the Mathematical Sciences Research Institute at Berkeley in June 1985, R. J. Archbold asked two specific questions related to this problem:

1. For a non-type I factor state of $\mathbf{B}$ can we get a factor state extension to $\mathbf{A}$ of type I, II, or III?

2. If φ is a factor state of $\mathbf{B}$, and if $S(\varphi)$ denotes the set of all extensions of φ to a state of $\mathbf{A}$, does the set of extreme points of $S(\varphi)$ contain a factor state of $\mathbf{A}$?

The Sakai extension technique mentioned before sheds very little light on Questions 1 and 2. In order to make progress on these questions, the development of new factor state extension techniques is therefore necessary. Woronowicz [**10**] has found a method which extends each factor state of a C^*-algebra $\mathbf{B}$ to a *pure state* of $\mathbf{B} \otimes \mathbf{B}^{\mathrm{op}}$ where $\mathbf{B}^{\mathrm{op}}$ is the C^*-algebra obtained from $\mathbf{B}$ by reversing its multiplication, and $\otimes$ denotes the *projective* C^*-tensor product. This is the best possible example of an affirmative answer to question 2 and to the type I conclusion of question 1. In addition to this, Tsui [**9**] has shown that if $\mathbf{B}$ is the commutant in

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a C^*-algebra $\mathbf{A}$ of a finite-dimensional C^*-subalgebra of $\mathbf{A}$, then Question 2 has an affirmative answer; in fact, *every* extreme point of $S(\varphi)$ is a factor state in this case.

In this paper, we add to this small list of examples one other by considering the unique, normalized trace on the Choi algebra, a factor state of type II_1, and extending it to a pure state of the Cuntz algebra O_2. While this result is admittedly very special, the technique involved makes use of the combinatorial features of the group structure of the Choi algebra–Cuntz algebra embedding, and describes how factor state extensions of the trace can be obtained by varying the way in which this group structure is realized spatially. The hope therefore is that a general group-theoretic approach to factor state extensions can be developed to eventually generate examples which illustrate the other possibilities that occur in Question 1. In that spirit, we close this introduction by posing the following problem: Find type II_∞ and type III_λ, $0 \leq \lambda \leq 1$, $\lambda \neq 1/2$, factor state extensions to O_2 of the trace on the Choi algebra (type II_1 factor states of O_2 are, of course, not possible since O_2 is simple and stable). It was pointed out by C. Lance that the $\mathrm{III}_{1/2}$-factor state extension to O_2 of the trace on the CAR C^*-subalgebra of D. Evans [**5**] is also an extension of the trace on the Choi algebra. (See Example 4.2 in [**1**].)

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2. A II_1-factor state of the C^*-algebra considered by M. D. Choi. Let $\mathbf{B}$ be the C^*-algebra generated by two unitary operators U, V on an infinite-dimensional Hilbert space such that $U^2 = V^3 = I$, $E + UEU = 1$, and $E + VEV^2 + V^2EV = 1$ for some orthogonal projection E. No projections F in the C^*-algebra generated by U, V satisfies the above equations. $\mathbf{B}$ is called the Choi algebra. According to the remark after Theorem 2.8 in [**2**], $\mathbf{B}$ has a unique tracial state which is called the trace in this article. For the sake of giving a complete exposition we provide proofs for the trace to be a nonhyperfinite II_1-factor state. It is shown in Corollary 2.7 in [**2**] that $\mathbf{B}$ is *-isomorphic to the reduced C^*-algebra of a discrete free group G on two generators u, v with $u^2 = v^3 = e$, the identity element in G.

LEMMA 2.1. *Every element in G, except e, has an infinite conjugacy class.*

PROOF. Let w be an arbitrary element in G other than e. w can be viewed as a word composed of u's and v's in reduced form. Suppose that it both begins and ends with u (or v, or v^{-1} respectively). Then $\{(uv)^k w (v^{-1}u)^k | k = 1, 2, \dots\}$ (or $\{(vu)^k w (uv^{-1})^k | k = 1, 2, \dots\}$ resp.) is an infinite set contained in the conjugacy class of w. Suppose that w begins with u and ends with v (or v^{-1} respectively). Then $z = v^{-1}wv$ (or $z = vwv^{-1}$ resp.) both begins and ends with v^{-1} (or v resp.). Hence z has an infinite conjugacy class, and so does w. The remaining case where w begins with v (or v^{-1}) and ends with u can be handled in a similar manner. Q.E.D.

Due to the work of Murray and von Neumann (Proposition 5 of III.7.6 in [**4**]) we know that the reduced von Neumann algebra $\mathbf{R}$ of G is a II_1-factor which is the weak-operator closure of $\mathbf{B}$. The unique normalized trace on $\mathbf{R}$, hence on $\mathbf{B}$, is a vector state induced by a cyclic vector χ_e, the characteristic function of the identity element e in G. Next, we show that $\mathbf{R}$ mentioned above is nonhyperfinite. It is sufficient to show $\mathbf{R}$ does not have property Γ (for definition see Definition 2 of III.7.7 in [**4**]). By Lemma 12 of III.7.7 in [**4**], we need only show

LEMMA 2.2. *The group G in Lemma* 2.1 *contains a subset X such that $X \cup uXu = G\backslash\{e\}$ and X, vXv^{-1}, $v^{-1}Xv$ are pairwise disjoint.*

PROOF. Let X consist of all reduced words in G beginning with u. If $w \in G\backslash X$ and $w \neq e$, then $z = uwu$ is in X and hence $w = uzu \in uXu$. vXv^{-1} (or $v^{-1}Xv$ respectively) consists of reduced words in G beginning with v (or v^{-1} respectively) and hence X, vXv^{-1}, $v^{-1}Xv$ are pairwise disjoint. Q.E.D.

Therefore we have shown

PROPOSITION 2.3. *The unique normalized trace of* $\mathbf{B}$ *is a nonhyperfinite* II_1-*factor state.*

3. Irreducible representations of the Cuntz algebras O_2 on $l^2(G)$. Let G be the free group on two generators u, v with $u^2 = v^3 = e$ as in §2. In this section we construct subsets F of G satisfying the following two sets of conditions:

(3.1) $F \cap (uF) = F \cap (vF) = \varnothing, \quad F \cup (uF) = G, \quad (vF) \cup (v^2F) = uF,$

(3.2) there exists a sequence of words in $u, v, \{z_i\}$, such that $\bigcap(z_iF) = \{e\}$.

Firstly we show how an irreducible representation of O_2 on $l^2(G)$ can be induced from such a subset F.

THEOREM 3.1. *Let F be a subset of G satisfying both conditions* (3.1) *and* (3.2). *Then there is an irreducible representation of O_2 on $l^2(G)$.*

PROOF. Let L be the left regular representation of G on $l^2(G)$. Let Lu, Lv be denoted by U, V respectively, and note that $U^2 = V^3 = I$. Let E be the projection of $l^2(G)$ onto the subspace generated by words in F. By condition (3.1), $E + U^*EU = I$, $E + V^*EV + VEV^* = I$. By Theorem 2.6 in [**2**], the C^*-algebra generated by U, V and E, $C^*(U, V, E)$, is isomorphic with O_2. In the rest of this section we identify O_2 with $C^*(E, U, V)$. For any two reduced words w_1, w_2 in u, v, we let χ_{w_i} denote the characteristic function of $\{w_i\}$, $i = 1, 2$, and, for any operator T in $\mathbf{B}$ $(l^2(G))$, the C^*-algebra of all bounded operators on $l^2(G)$, let $\langle T\chi_{w_1}, \chi_{w_2}\rangle$ be denoted by T_{w_1,w_2}. If $T \in O_2'$, the commutant of O_2 in $\mathbf{B}(l^2(G))$, then $T_{w_1,w_2} = T_{ww_1,ww_2}$ for all words w in u, v. In fact,

$$\begin{aligned} T_{ww_1,ww_2} &= \langle T\chi_{ww_1}, \chi_{ww_2}\rangle = \langle TL\chi_w(\chi_{w_1}), L\chi_w(\chi_{w_2})\rangle \\ &= \langle (L\chi_w)^*T(L\chi_w)(\chi_{w_1}), \chi_{w_2}\rangle = \langle T\chi_{w_1}\chi_{w_2}\rangle = T_{w_1,w_2} \end{aligned}$$

for any word w in u, v, and $T \in O_2'$. In particular, $T_{w,w} = T_{e,e}$ for all words w in u, v. Finally, for two reduced words $w_1 \neq w_2$, $T_{w_1,w_2} = T_{e,w_1^{-1}w_2}$ and let $w_1^{-1}w_2 = w \neq e$. Then there is a positive integer n such that $w \notin z_nF$ whereas $e \in z_nF$, i.e., $z_n^{-1}w \notin F$ and $z_n^{-1} \in F$. Hence $T_{e,w} = T_{z_n^{-1},\, z_n^{-1}w} = \langle T\chi_{z_n^{-1}}, \chi_{z_n^{-1}w}\rangle = 0$, for $\chi_{z_n^{-1}} \in E(l^2(G)) \Rightarrow T\chi_{z_n^{-1}} \in E(l^2(G))$, and $\chi_{z_n^{-1}w}$ is in the orthogonal complement of $E(l^2(G))$. This proves that O_2 acts irreducibly on $l^2(G)$. Q.E.D.

As is stated in the proof of Theorem 3.1, the left regular representation together with a choice of a subset F gives rise to an irreducible representation π of O_2, which is also equivalent to the GNS representation induced by the vector state ω_{χ_e}, the trace on $\mathbf{B}$, and $\omega_{\chi_e} \circ \pi$ is regarded as an extension of the trace on $\mathbf{B}$ to O_2. The corollary below follows.

COROLLARY 3.2. *The trace on* $\mathbf{B}$ *has a pure state extension to* O_2.

Throughout the rest of this section the complement of a subset S of G will be denoted by S^c, and both δ and σ will be either 1 or 2. In the proposition below we give a slightly different characterization of a subset F of G satisfying condition (3.1).

PROPOSITION 3.3. *Any subset F of G satisfying condition* (3.1) *also satisfies the following*:

(1) $v^\delta F \subseteq F^c$ *for* $\delta = 1, 2$ *and* $uF = F^c$.

(2) $(uv^\delta)F \subseteq F$ *for* $\delta = 1, 2$.

(3) $(v^\delta u)F^c \subseteq F^c$ *for* $\delta = 1, 2$.

(4) *If* $w \in F$, *then exactly one of* $(vu)w$ *and* $(v^2u)w$ *is in* F.

(4)′ *If* $w \in F^c$, *then exactly one of* vw *and* v^2w *is in* F.

In particular, conditions (4) *and* (4)′ *are equivalent under the assumption* $uF = F^c$. *Furthermore the set of conditions* (1) *and* (4), *denoted by* (3.3), *is equivalent to* (3.1).

PROOF. (1) obviously follows from (3.1).

Since $v^\delta F \subseteq F^c = uF$, it follows that $(uv^\delta)F \subseteq uF^c = F$. Similarly $(v^\delta u)F^c = (v^\delta u)uF = v^\delta F \subseteq F^c$. Then (2) and (3) are shown.

Let $w \in F^c = (vF) \cup (v^2F)$. If $w \in vF$, then $v^2w \in v^2(vF) = F$ and $vw \in v^2F \subseteq F^c$. If $w \in v^2F$, then $vw \in F$ and $v^2w \in vF \subseteq F^c$. Thus (4)′ is true.

Now we assume $uF = F^c$. Replacing w by uw in (4)′ one gets (4) and symmetrically replacing w by uw in (4) one gets (4)′.

Next we assume that a subset F of G satisfies (3.3). It is obvious that $F \cap (uF) = \varnothing = F \cap (vF)$, $F \cup (uF) = G$, and $(v^\delta F) \subseteq (uF)$. We need only show $(uF) \subseteq (vF) \cup (v^2F)$, which follows from $uF = F^c$ and condition (4). Q.E.D.

Here we set up some notation for the rest of this section. Let A be the set of all words of the form

$$(uv^{\delta_n})(uv^{\delta_{n-1}})\cdots(uv^{\delta_1}) \quad \text{or} \quad e,$$

where δ_i is either 1 or 2 for $1 \leq i \leq n$. It follows from (2) in Proposition 3.3 that if F satisfies (3.1), and $w \in F$, then xw must be in F for all x in A. Next, we let $\theta = \{n_i\}$ be a strictly increasing sequence of positive integers. For any (fixed) $w \in G$, we denote by $S(\theta, w)$ the set of all words of the form

$$(v^\delta u)^{n_p}\cdots(v^2u)^{n_2}(vu)^{n_1}w, \qquad p = 0, 1, 2, \ldots,$$

where δ is 2 for even p and δ is 1 for odd p. The length of a reduced word in u, v is the number of characters in that word.

PROPOSITION 3.4. *Suppose that a subset F of G satisfies* (3.1) *and* $S(\theta, v) \subseteq F$. *Then v is the only word w in u, v such that* $S(\theta, w) \subseteq F$.

In order to prove Proposition 3.4 we need the following lemmas in which F is assumed to satisfy (3.1) and $S(\theta, v) \subseteq F$.

LEMMA 3.5. *Let $p = 0, 1, 2, \ldots$, δ be 2 for even p and 1 for odd p, and $\bar{\delta} = 3-\delta$. Then the words of the form*

$$(v^{\bar{\delta}}u)^k(v^{\delta}u)^{n_p}\cdots(vu)^{n_1}v, \qquad 0 \le k < n_{p+1},$$

are in F.

PROOF. Observe that $w = (v^{\bar{\delta}}u)^{n_{p+1}}(v^{\delta}u)^{n_p}\cdots(vu)^{n_1}v$ is in F. Then straightforward applications of condition (2) in Proposition 3.3 to w will yield the desired result. Q.E.D.

LEMMA 3.6. *Let δ be 2 for even p and 1 for odd p, $p = 1, 2, \ldots$. Consider any word $w = (v^{\delta}u)^{m_p}\cdots(v^2u)^{m_2}(vu)^{m_1}v$, where $m_j \ge 0$, $1 \le j \le p$, and $m_i \ne n_i$ for at least one i with $1 \le i \le p-1$, or $m_i = n_i$ for all $1 \le i \le p-1$ and $m_p > n_p$. Words thus obtained are not in F.*

PROOF. Let k be the smallest integer such that $m_k \ne n_k$ in the word described above, and denote it by w_0. Suppose $w_0 \in F$. We will deduce a contradiction from this supposition. In what follows we let σ, ρ be 1 or 2 and $\bar{\rho} = 3-\rho$, $\bar{\sigma} = 3-\sigma$.

Case 1. Suppose $m_k < n_k$. This implies that $k < p$. By Lemma 3.5 we know that both $w_1(v^{\sigma}u)^{m_k}(v^{\bar{\sigma}}u)^{n_{k-1}}\cdots(vu)^{n_1 v}$ and $(v^{\sigma}u)w_1$ are in F. Several applications of (2) in Proposition 3.3 to w_0 show that $(v^{\bar{\sigma}}u)w_1$ is also in F. This is a contradiction to (4) in Proposition 3.3.

Case 2. Suppose $m_k > n_k$. Applying (2) in Proposition 3.3 to w_0 a number of times, one sees that both $s = (v^{\rho}u)^{n_k}\cdots(vu)^{n_1}v$ and $(v^{\rho}u)s$ are in F. By Lemma 3.5 $(v^{\bar{\rho}}u)s$ is also in F. Again this is a contradiction to (4) in Proposition 3.3. Q.E.D.

LEMMA 3.7. *Let $\{m_1, m_2, \ldots, m_k\}$ be a nonempty set of positive integers. Then words of the form*

$$(v^{\sigma}u)^{n_p}\cdots(v^2u)^{n_2}(vu)^{n_1}(v^2u)^{m_k}\cdots(v^{\delta}u)^{m_1}v \quad \text{with } p \ge 2,$$

where σ is 2 for even p and 1 for odd p, and δ is 1 for even k and 2 for odd k, are not in F.

PROOF. Since $\{n_i\}$ is an increasing sequence and $p \ge 2$, the adjoint sequence $\{m_1, m_2, \ldots, m_k, n_1, n_2, \ldots, n_p\}$ must differ from $\{n_1, \ldots, n_k, \ldots, n_{p+k}\}$ at the $(k+1)$st term. If k is odd, the words described in Lemma 3.7 are of the form $(v^{\sigma}u)^{n_p}\cdots(vu)^{n_1}(v^2u)^{m_k}\cdots(v^2u)^{m_1}v$ and thus is not in F by Lemma 3.6. If k is even, the words described in Lemma 3.7 are of the form

$$(v^{\sigma}u)^{n_p}\cdots(vu)^{n_1}(v^2u)^{m_k}\cdots(vu)^{m_1}v$$

in which $n_1 \ne n_{k+1}$. Hence they are not in F by Lemma 3.6. Q.E.D.

REMARK 3.8. Actually Lemmas 3.5, 3.6, and 3.7 remain true for those subsets F of G satisfying (2) and (4) in Proposition 3.3 and $S(\theta, v) \subseteq F$.

LEMMA 3.9. *No reduced words beginning with v or v^2 and ending with u or v^2 can be in F.*

PROOF. From Lemma 3.5 we see $v \in F$ and hence $v^2v = e \in F^c$, $v^2 = v\cdot v \in F^c$ and $u = u\cdot e \in F$, by (1) in Proposition 3.3. A reduced word, beginning with v^{δ} and ending with u, is of even length, say $2n$, $n \ge 0$. If such a word is in F, then applying (2) in Proposition 3.3 n times, one gets $e \in F$, a contradiction. Hence such a word is not in F. A reduced word, beginning with v^δ and ending with v^2, is of odd length, say $1+2n$, $n \geq 0$. If such a word is in F, then applying (2) in Proposition 3.3 n times, one gets $v^2 \in F$, a contradiction. Hence such a word is not in F. Q.E.D.

PROOF OF PROPOSITION 3.4.

Case 1. Let w be a reduced word beginning with v^σ, $\sigma = 1$ or 2, but different from v.

Consider $w_1 = (v^2u)^{n_2}(vu)^{n_1}w$ which is not in F if w ends with v^2 or u by Lemma 3.9. While w ends with v, w_1 cannot be in F by Lemma 3.7. Thus $S(\theta, w) \not\subseteq F$.

Case 2. Let w be a reduced word beginning with u of length p.

Consider the word $w_2 = (v^2u)^{n_{2p}} \cdots (vu)^{n_1}w$. In its reduced form w_2 begins with v^2 and ends with either u or v^2, hence not in F, or v. When it ends with v, it will be like words in Lemma 3.6, hence not in F. Thus $S(\theta, w) \not\subseteq F$. Q.E.D.

PROPOSITION 3.10. *Suppose that a subset F of G satisfies* (3.1) *and* $S(\theta, v) \subseteq F$. *Then F satisfies* (3.2) *as well.*

PROOF. Let $z_i = v^2(uv^2)^{n_1}(uv)^{n_2} \cdots (uv^\sigma)^{n_i}$ for $i = 1, 2, \ldots$, where $\sigma = 2$ if i is odd and $\sigma = 1$ if i is even. We need only show $\bigcap_i(z_iF) = \{e\}$. Since $z_i^{-1} = (v^{\bar\sigma}u)^{n_i} \cdots (vu)^{n_1}v$, where $\bar\sigma = 3 - \sigma$, is in F for all $i = 1, 2, \ldots$, it follows that $\{e\} \subseteq \bigcap_i(z_iF)$. The presence of a word w in $\bigcap_i(z_iF)$ implies that $z_i^{-1}w$ is in F for all $i = 1, 2, \ldots$, which is the same as $S(\theta, vw) \subseteq F$. By Proposition 3.4 we have $vw = v$ and hence $w = e$. Thus $\{e\} = \bigcap_i(z_iF)$. Q.E.D.

Following from Proposition 3.10 and the remark preceding Proposition 3.4, it is only natural to try to construct a subset F satisfying (3.1) and (3.2) by the procedure below:

Given an increasing sequence $\theta = \{n_i\}$ of positive integers, let F_θ be the subset of G consisting of words in the form of xy, with $x \in A$ and $y \in S(\theta, v)$. (For the definition of A and $S(\theta, v)$, see the remark before Proposition 3.4.) It is obvious that $S(\theta, v) \subseteq F_\theta$, because $e \in A$. Thus $v \in F_\theta$ and $u = (uv^2)v \in F_\theta$. Many examples of such subsets F_θ can be obtained. In a forthcoming paper we will show that all these uncountable irreducible representations of O_2 constructed from θ's with different tails are inequivalent. By Proposition 3.10 only the validity of (3.1) for F_θ remains to be shown.

PROPOSITION 3.11. *F_θ satisfies* (3.1).

PROOF. By Proposition 3.3 it is sufficient to show that F_θ satisfies conditions (1) and (4) in Proposition 3.3. Actually F_θ is the union of the following disjoint subsets:

$F_1 = \{\text{elements in Lemma 3.5}\}$,

$F_2 = \{(uv^{\delta_n}) \cdots (uv^{\delta_1})(uv^\delta u)(v^{\bar\delta}u)^k(v^\delta u)^{n_p} \cdots (v^2u)^{n_2}(vu)^{n_1}v$
with $0 \leq k < n_{p+1}$ for some $n, p \geq 0$, where $\delta_i = 1$ or 2 for $i = 1, \ldots, n$, and $\delta = 2$ if p is even and $\delta = 1$ if p is odd, $\bar\delta = 3 - \delta\}$,

$F_3 = \{(uv^{\delta_n}) \cdots (uv^{\delta_1})(uv^2)$ or $(uv^{\delta_n}) \cdots (uv^{\delta_1})u$,
$n = 0, 1, 2, \ldots$, where $\delta_i = 1$ or 2 for $i = 1, \ldots, n\}$.

First of all, we check that conditions (2) and (4) in Proposition 3.3 hold for F_θ, and then Lemmas 3.5, 3.6, and 3.7 can apply to F_θ. Finally we will show that condition (1) in Proposition 3.3 holds for F_θ.

Let w be in F_1 and $w = (v^{\bar{\delta}}u)^k(v^\delta u)^{n_p}\cdots(vu)^{n_1}v$, $0 \le k < n_{p+1}$, $\bar{\delta} = 3 - \delta$. Thus $(v^{\bar{\delta}}u)w \in F_\theta$ and $(v^\delta u)w \notin F_j$ for $j = 1, 2, 3$, hence w is not in F_θ. For $k > 0$, $(uv^\delta)w \in F_1$ and $(uv^{\bar{\delta}})w \in F_2$. For $k = 0$, $(uv^{\bar{\delta}})w \in F_1$ and $(uv^\delta)w \in F_2$.

Let w be in F_2. Thus $(v^{\bar{\delta}_n}u)w \in F_2$, $(v^{\delta_n}u)w \notin F_\theta$ if $n \neq 0$, and $(v^\delta u)w \in F_1$, $(v^{\bar{\delta}}u)w \notin F_\theta$ if $n = 0$, by the description of F_1, F_2, F_θ. It is easy to see that $(uv^\delta)w \in F_2$ for $\delta = 1, 2$.

Now let w be in F_3. It is obvious that $(uv^\delta)w \in F_3$ for $\delta = 1, 2$. $(v^{\bar{\delta}_n}u)w \in F_3$ and $(v^{\delta_n}u)w \notin F_\theta$ if $n > 0$. Otherwise $(v^2u)(uv^2) \in F_\theta$, $(vu)(uv^2) \notin F_\theta$ and $(v^2u)u \notin F_\theta$, $(vu)u \in F_\theta$. Hence condition (2) and (4) in Proposition 3.3 hold for F_θ.

Next we check that condition (1) in Proposition 3.3 holds for F_θ. Firstly we show $v^\delta F_\theta \subseteq F_{\theta^c}$ for $\delta = 1, 2$.

Let $w \in F_3$. Then $v^\delta w \in F_\theta^c$, for $v^\delta w$ in its reduced form begins with v^δ and ends with v^2 or u, hence is not in F_θ by the description of F_1, F_2, F_3.

Let $w \in F_2$. Then $v^\delta w$ is like a word described in Lemma 3.6 and hence is not in F_θ.

Let $w \in F_1$. One checks by inspection that for $\delta = 1$ or 2, $v^\delta w \notin F_j$, $j = 1, 2, 3$, and hence $v^\delta w \notin F_\theta$ for $\delta = 1, 2$.

Secondly we check $uF_\theta \subseteq F_{\theta^c}$.

Let $w \in F_1$. Then by inspection uw does not belong to F_j for $j = 1, 2, 3$ and hence $uw \in F_{\theta^c}$.

For any word w in F_3, except $w = u$, uw begins with v^δ, $\delta = 1$ or 2, and ends with v^2 or u, hence not in F_θ. In case $w = u$, $uw = e$ is not in F_θ.

For any word w in F_2, except $w = v$, uw is like a word described in Lemma 3.6, hence not in F_θ. In case $w = v$, uv is not in F_θ.

Finally, we check $uF_\theta^c \subseteq F_\theta$.

Case 1. w begins with v^δ, $\delta = 1$ or 2.

If w ends with u or v^2, then $uw \in F_3$. Otherwise, w must be of the form $(v^\delta u)^{r_s}\cdots(v^2u)^{r_2}(vu)^{r_1}v$ for some $s > 0$ and $r_i \ge 0$ for $i = 1, \ldots, s$. We note that one of r_i's must be nonzero, for otherwise $w = v \in F_\theta$. Actually we may assume that $r_i > 0$ for $2 \le i \le s$. If $r_1 = 0$, then $s \ge 2$ and uw is in F_2, as an element of the general form in F_2 with $p = k = 0$. Suppose $r_1 \neq 0$. Since $w \notin F_\theta$, there is at least one i such that $r_i \neq n_i$. Let j be the smallest integer such that $r_j \neq n_j$. If $r_j > n_j$, then uw is in F_2, as an element of the general form in F_2 with $k = 0$, $p > 0$. If $r_j < n_j$, one sees that uw is in F_2 as an element of the general form with $k = r_j$, because $w \notin F_\theta$, $j < s$.

Case 2. w begins with u.

Then w must end with v and is of the form $u(v^\delta u)^{r_s}\cdots(v^2u)^{r_2}(vu)^{r_1}v$ for some $s \ge 0$ and $r_i \ge 0$ for $i = 1, \ldots, s$. We may assume $s > 0$, and one of the r_i's is nonzero, for otherwise $w = uv$ and $uw \in F_\theta$. Let w_1 be uw. Then w_1, in its reduced form beginning with v^δ and ending with v, cannot be in F_θ^c, for otherwise $uw_1 = w$ would be in F_θ by Case 1 above. Hence $w_1 = uw$ must be in F_θ. Q.E.D.

REFERENCES

1. R. J. Archbold and C. J. K. Batty, *Extensions of factorial states of C^*-algebras*, J. Funct. Anal. **63** (1985), 86–100.
2. M. D. Choi, *A simple C^*-algebra generated by two finite order unitaries*, Canad. J. Math. **31** (1979), 867–880.
3. J. Cuntz, *Simple C^*-algebra generated by isometries*, Comment. Math. Phys. **57** (1977), 173–185.
4. J. Dixmier, *Les algèbres d'opérateurs dans l'espace Hilbertien*, Gauthier-Villars, Paris, 1969.
5. D. E. Evans, *On O_n*, Publ. Res. Inst. Math. Sci. **16** (1980), 915–927.
6. R. Longo, *Solution of the factorial Stone-Weierstrass conjecture. An application of standard split W^*-inclusions*, Invent. Math. **76** (1984), 145–155.
7. S. Popa, *Constructing semiregular maximal abelian *-subalgebras in factors*, Invent. Math. **65** (1984), 157–161.
8. S. Sakai, Talk given in Wabash International Conference on Banach Spaces in 1973, unpublished.
9. S.-k. Tsui, *Central sequences associated with a state*, Proc. Amer. Math. Soc. **82** (1981), 76–81.
10. S. L. Woronowicz, *On the purification of factor states*, Comm. Math. Phys. **78** (1972), 221–235.

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