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A REFINEMENT OF SARKOVSKII'S THEOREM

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ABSTRACT. Let $f: R \to R$ be continuous. If f has an orbit of period n, the question of which other periods f must necessarily have was answered by A. N. Sarkovskii by giving a total ordering of the natural numbers, now called the *Sarkovskii ordering*. The ordering does not take into account the period types and examples show that depending on the type of the period other periods than those implied by the Sarkovskii ordering are present. Introducing the concepts of a periodic loop (a periodic orbit of a certain type) and infinite loop, we give a total ordering of loops and obtain, as a consequence, a refinement of the theorem of Sarkovskii.

1. Introduction. It is known that a four-periodic orbit of a continuous function $f: R \to R$ may imply a three-periodic orbit and hence an n-periodic orbit for every $n = 1, 2, \ldots$ [**3**, Theorem 3]. This orbit has, of course, not the same structure, or, as we shall say, is not of the same "type", as the four-periodic orbit that appears in the Sarkovskii ordering. Such and similar implications that are not accounted for in Sarkovskii's theorem compellingly draw attention to and exemplify a problem that may be called the "type-problem": "Given a positive integer n and an n-periodic orbit of a specified type, find, for every positive integer m, the types of m-periodic orbits that must exist". Sarkovskii's result gives only the (complete) answer to the restricted problem: "Given a positive integer n and an n-periodic orbit of any type, for which other integers m does there exist an m-periodic orbit of any type?" At this stage of knowledge the type-problem is an open problem of considerable complexity. To obtain a first result towards the solution of the type-problem, we have singled out from the set of $(n-1)!$ different types of n-periodic orbits that a continuous function can have a specific type and called it a loop. We also introduce the notion of infinite loop. By combining the total ordering of loops with the Sarkovskii ordering, we arrive at a refinement of Sarkovskii's theorem, Theorem (SR). The purpose of this paper is to prove Theorem (SR), stated in §3, and, to emphasize the importance of loops, strengthen a number of results that have recently appeared in the literature. In addition, since the notion of turbulence introduced by Block and Coppel in [**6**] and the notion of infinite loop are equivalent, a streamlined proof for this important characterization of turbulence is given. These results are contained in §6.

2. Definitions and notation. Let $f: R \to R$ be continuous and $x_0 \in R$. The orbit of x_0 under f is defined as the set $\{x : x = f^n(x_0),\ n = 0, 1, \ldots\}$, where, for

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every positive integer n, f^n is the nth iterate of f, $f^1 = f$, and $f^0(x_0) = x_0$. We shall write $x_n := f^n(x_0)$ for a given $x_0 \in R$ and call $x_1, x_2, \ldots$ the successors of x_0. A preorbit of a given $x_0 \in R$ is any (finite or infinite) sequence $x_0, x_{-1}, x_{-2}, \ldots$ such that $f(x_{-n}) = x_{-(n-1)}$ for all n for which x_{-n} is defined. The points $x_{-1}, x_{-2}, \ldots$ in any such sequence are called predecessors of x_0. A point c_0 is called critical if $f(c_0) = c_0$, i.e., a critical point of f is a fixed point of f. A periodic point x_0 of period $p > 1$ (p a positive integer) is a point for which the relations $f^p(x_0) = x_0$, $f^k(x_0) \neq x_0$, $1 \leq k < p$, hold. If x_0 is a periodic point of period p, its orbit is denoted by $(x_0, x_1, \ldots, x_{p-1})$. We shall denote the kth iterate of x_0 under the function f^m by x_k^m, $k = 0, 1, \ldots$. Thus $x_k^m := (f^m)^k(x_0) = x_{mk}$, and, in particular, $x_0^m = x_k^0 = x_0$ for all nonnegative integers k and m.

DEFINITION. Let $f\colon R \to R$ be continuous and $x_0 \in R$. f has a loop of order n if x_0 has a preorbit $(x_0, x_{-1}, \ldots, x_{-n})$ such that either

$$x_0 < x_{-n} < x_{-(n-1)} < \cdots < x_{-2} < x_{-1}$$

or

$$x_0 > x_{-n} > x_{-(n-1)} > \cdots > x_{-2} > x_{-1}.$$

f has an infinite loop if x_0 has an infinite preorbit $(x_0, x_{-1}, \ldots, x_{-n}, \ldots)$ such that either

$$x_0 < \cdots < x_{-n} < x_{-(n-1)} < \cdots < x_{-2} < x_{-1}$$

or

$$x_0 > \cdots > x_{-n} > x_{-(n-1)} > \cdots > x_{-2} > x_{-1}.$$

A loop of order $(n-1)$ is called an n-periodic loop if $x_0 = x_{-n}$.

We adopt the following concise notation: we say property $P(k)$ holds if f has a periodic orbit of period k. Thus $P(1), L(k), L(\infty)$ mean that f has a critical point, a period loop of period k, an infinite loop, respectively. Similarly, $P^n(k), L^n(k), L^n(\infty)$ shall mean that f^n has a k-periodic orbit, k-periodic loop, an infinite loop, respectively.

3. Sarkovskii's theorem and theorem (SR). Using the notation introduced in §2, Sarkovskii's theorem and our refinement read as follows.

THEOREM (SARKOVSKII). *Let $f\colon R \to R$ be continuous. Then*

$$\begin{aligned}
&P(3) \Rightarrow P(5) \Rightarrow P(7) \Rightarrow \cdots \Rightarrow \\
&P(2\cdot 3) \Rightarrow P(2\cdot 5) \Rightarrow P(2\cdot 7) \Rightarrow \cdots \Rightarrow \\
&P(2^2\cdot 3) \Rightarrow P(2^2\cdot 5) \Rightarrow P(2^2\cdot 7) \Rightarrow \cdots \Rightarrow \\
&\cdots \qquad\qquad \Rightarrow \\
&P(2^3) \Rightarrow P(2^2) \Rightarrow P(2) \Rightarrow P(1).
\end{aligned}$$

THEOREM (SR). *Let $f\colon R \to R$ be continuous. Then*

$$\begin{aligned}
&L(\infty) \Rightarrow \cdots \Rightarrow L(5) \Rightarrow L(4) \Rightarrow L(3) \Leftrightarrow \\
&P(3) \Rightarrow P(5) \Rightarrow P(7) \Rightarrow \cdots \Rightarrow \\
&L^2(\infty) \Rightarrow \cdots \Rightarrow L^2(5) \Rightarrow L^2(4) \Rightarrow L^2(3) \Leftrightarrow \\
&P(2 \cdot 3) \Rightarrow P(2 \cdot 5) \Rightarrow P(2 \cdot 7) \Rightarrow \cdots \\
&L^{2^2}(\infty) \Rightarrow \cdots \Rightarrow L^{2^2}(5) \Rightarrow L^{2^2}(4) \Rightarrow L^{2^2}(3) \Leftrightarrow \\
&P(2^2 \cdot 3) \Rightarrow P(2^2 \cdot 5) \Rightarrow P(2^2 \cdot 7) \Rightarrow \cdots \\
&\cdots \\
&\cdots \Rightarrow P(2^3) \Rightarrow P(2^2) \Rightarrow P(2) \Rightarrow P(1).
\end{aligned}$$

4. Elementary lemmas. It follows from the definition of a periodic loop that every three-periodic orbit is a three-periodic loop and that an $(n+1)$-periodic loop implies the existence of a loop of order n.

LEMMA 4.1. *If f has a critical point c_0 such that $c_0 < c_{-2} < c_{-1}$, then f has an infinite loop satisfying $c_0 < \cdots < c_{-n} < \cdots < c_{-2} < c_{-1}$. The same statement holds with all inequalities reversed.*

PROOF. Since $f(c_{-2}) = c_{-1}$ and $c_0 < c_{-1}$, there exists $c_{-3} \in (c_0, c_{-2})$. Repeating this argument establishes the lemma.

LEMMA 4.2. *If f has a critical point c_0 such that $c_{-1} < c_{-3} < c_0 < c_{-2}$, then c_0 has an infinite preorbit satisfying $c_{-1} < c_{-3} < \cdots < c_0 < \cdots < c_{-4} < c_{-2}$. In particular, f^2 has two infinite loops. The same statement holds with all inequalities reversed.*

PROOF. Since $f(c_{-2}) = c_{-1} < c_{-3}$ and $c_0 > c_{-3}$, there exists $c_{-4} \in (c_0, c_{-2})$, and since $f(c_{-3}) = c_{-2} > c_{-4} > c_0$, there exists $c_{-5} \in (c_{-3}, c_0)$. Repeating this argument proves the lemma.

LEMMA 4.3. $P^{2^k}(n) \Leftrightarrow P(2^k \cdot n)$, $n, k = 1, 2, \ldots$.

PROOF. It suffices to show that $P^2(n) \Leftrightarrow P(2 \cdot n)$. If $(x_0, x_1, \ldots, x_{2n-1})$ is a $2n$-periodic orbit of f, then $(x_0^2, x_1^2, \ldots, x_{n-1}^2)$ is an n-periodic orbit of f^2. Hence $P(2 \cdot n) \Rightarrow P^2(n)$. If $(x_0^2, x_1^2, \ldots, x_{n-1}^2)$ is an n-periodic orbit of f^2, we consider the set $\{x_0, x_1, \ldots, x_{2n-1}\}$, where $x_{2n} = x_n^2 = x_0^2 = x_0$. If $x_0 \neq x_k$, $k = 1, 2, \ldots, 2n-1$, then $C = (x_0, x_1, \ldots, x_{2n-1})$ is a $2n$-periodic orbit of f. Otherwise, there is a smallest odd k, $1 < k < 2n$, such that $x_0 = x_k$, i.e., x_0 is an odd-periodic point of f. But then, by Sarkovskii's theorem, f has periodic orbits of every even period and, therefore, in particular, a $2n$-periodic orbit. Hence $P^2(n) \Rightarrow P(2 \cdot n)$ and the proof of the lemma is complete.

5. Principal results. Let $C = (x_0, x_1, \ldots, x_{n-1})$ be any n-periodic orbit of f. We define the subsets

$$C^+ = \{x_i \in C\colon x_{i+1} > x_i\}, \quad C^- = \{x_i \in C\colon x_{i+1} < x_i\},$$
$$D^+ = \{x_i \in C\colon x_{i+2} > x_{i+1} > x_i\}, \quad D^- = \{x_i \in C\colon x_{i+2} < x_{i+1} < x_i\}.$$

The sets C^+ and C^- are nonempty since $\min C \in C^+$ and $\max C \in C^-$. Letting further $a_0^+ = \min C^+$ $(= \min C)$, $b_0^+ = \max C^+$, $a_0^- = \min C^-$, and $b_0^- = \max C^-$ $(= \max C)$, it is clear that either $a_0^+ \leq b_0^+ < a_0^- \leq b_0^-$ or $a_0^+ < a_0^- < b_0^+ < b_0^-$.

THEOREM 5.1. *If* $a_0^+ \leq b_0^+ < a_0^- \leq b_0^-$ *and* $D^+ \cup D^- \neq \emptyset$, *then* f *has a critical point* c_0 *such that* f^2 *has two infinite loops* $(d_0^2, d_{-1}^2, d_{-2}^2, \dots)$ *and* $(c_0^2, c_{-1}^2, c_{-2}^2, \dots)$ *satisfying*

$$d_{-1}^2 < d_{-2}^2 < \cdots < d_0^2 = c_0 = c_0^2 < \cdots < c_{-2}^2 < c_{-1}^2.$$

In particular, $L^2(\infty)$ *holds.*

PROOF. It is sufficient to assume that $D^+ \neq \emptyset$. Then, if we let $\beta_0^+ = \max D^+$, we have

$$a_0^+ \leq \beta_0^+ < \beta_1^+ \leq b_0^+ < a_0^- \leq \beta_2^+ \leq b_0^-,$$

and conclude the existence of a critical point c_0 and a predecessor c_{-1} such that

$$a_0^+ \leq \beta_0^+ < c_{-1} < \beta_1^+ \leq b_0^+ < c_0 < a_0^- \leq b_0^-.$$

We now consider the set $E^- = \{x_i \in C^- : x_{i+1} < c_{-1}\}$. E^- is nonempty since $a_{n-1}^+ \in C^-$ and $a_n^+ = a_0^+ < c_{-1}$. Letting $r_0^- = \min E^-$, we have

$$a_0^+ \leq r_1^- < c_{-1} < b_0^+ < c_0 < a_0^- \leq r_0^- \leq b_0^-.$$

This shows that, since $c_0 > c_{-1}$ and $r_1^- < c_{-1}$, there exists a predecessor c_{-2} such that

$$a_0^+ < c_{-1} < b_0^+ < c_0 < c_{-2} < r_0^- \leq b_0^-.$$

Our construction implies that

(i) if $x_i \in C^+$ and $x_i > c_{-1}$, then $x_{i+1} \in C^-$;

(ii) if $x_i \in C^-$ and $x_i < r_0^-$, then $x_{i+1} > c_{-1}$.

Hence, there is an $x_i \in C^+$, $x_i > c_{-1}$ such that $x_{i+1} \geq r_0^-$. For otherwise we would have $b_i^+ \in (c_{-1}, r_0^-)$ for all i, contradicting the fact that C is the orbit of b_0^+ (thus $b_i^+ = a_0^+$ for some $i > 1$). We now choose $\delta_0^+ \in C^+$ such that $\delta_0^+ > c_{-1}$ and $\delta_1^+ \geq r_0^-$ to obtain

$$a_0^+ < c_{-1} < \delta_0^+ \leq b_0^+ < c_0 < c_{-2} < r_0^- \leq \delta_1^+ \leq b_0^-.$$

But this implies that we may choose a predecessor c_{-3} in the interval (c_{-1}, δ_0^+), and hence that c_0 and its predecessors c_{-1}, c_{-2}, and c_{-3} satisfy the inequality $c_{-1} < c_{-3} < c_0 < c_{-2}$. Appeal to Lemma 4.2 completes the proof.

THEOREM 5.2. *If* $a_0^+ < a_0^- < b_0^+ < b_0^-$, *there exist critical points* d_0, c_0 *of* f *and two infinite loops* $(d_0, d_{-1}, d_{-2}, \dots)$ *and* $(c_0, c_{-1}, c_{-2}, \dots)$ *of* f *satisfying* $d_{-1} < d_{-2} < \cdots < d_0 \leq c_0 < \cdots < c_{-2} < c_{-1}$. *In particular,* $L(\infty)$ *holds.*

PROOF. We note first that there is $\alpha_0^- \in C^-$ and $\beta_0^+ \in C^+$ such that

(i) $a_0^+ < a_0^- \leq \alpha_0^- < \beta_0^+ \leq b_0^+ < b_0^-$;

(ii) if $x_i \in C$, then $x_i \leq \alpha_0^-$ or $x_i \geq \beta_0^+$;

(iii) if $x_i \in C$ and $\alpha_0^- < x_i \leq b_0^+$, then $x_i \in C^+$;

(iv) if $x_i \in C$ and $b_0^+ < x_i \leq b_0^-$, then $x_i \in C^-$.

We now show that there are predecessors c_{-1} and c_{-2} of the critical point $c_0 \in (\alpha_0^-, \beta_0^+)$ that satisfy the inequality $c_0 < c_{-2} < c_{-1}$. The set $A^- = \{x_i \in C^- : x_i > b_0^+ \text{ and } x_{i+1} \leq \alpha_0^-\}$ is nonempty (otherwise $\beta_n^+ \geq \beta_0^+$ for all integers $n \geq 0$, a contradiction). Let $r_0^- = \min A^-$. We have $r_0^- > b_0^+$ and observe that the set $A^+ = \{x_i \in C^+ : \beta_0^+ \leq x_i \leq b_0^+ \text{ and } x_{i+1} \geq r_0^-\}$ is nonempty (since otherwise β_n^+

will satisfy $\beta_0^+ \leq \beta_n^+ < r_0^-$ for $n \geq 0$, a contradiction). We choose any $y_0^+ \in A^+$ and have

$$\alpha_0^- < c_0 < \beta_0^+ \leq y_0^+ \leq b_0^+ < r_0^- \leq b_0^- .$$

Hence

$$c_0 < c_{-2} < y_0^+ \leq b_0^+ < c_{-1} < r_0^- \leq b_0^- ,$$

where the existence of c_{-1} follows from $b_1^+ > c_0$ and $r_1^- < c_0$ and that of c_{-2} from $c_0 < c_{-1}$ and $y_1^+ \geq r_0^- > c_{-1}$. The infinite loop $(c_0, c_{-1}, c_{-2}, \dots)$ satisfying $c_0 < \cdots < c_{-2} < c_{-1}$ follows from Lemma 4.1. An analogous procedure locates a critical point d_0 and predecessors d_{-1}, d_{-2} such that $d_{-1} < d_{-2} < d_0 \leq c_0$, and hence an infinite loop $(d_0, d_{-1}, d_{-2}, \dots)$ satisfying $d_{-1} < d_{-2} < \cdots < d_0 \leq c_0$. This completes the proof.

THEOREM 5.3. *If f has a loop of order $n \geq 3$, then f has two distinct n-periodic loops. In particular, $L(n)$ holds.*

PROOF. Let $(x_0, x_{-1}, \dots, x_{-n})$ be a loop of order $n \geq 3$ of f such that $x_0 < x_{-n} < \cdots < x_{-2} < x_{-1}$. Since there is a critical point $c_0 \in (x_{-2}, x_{-1})$, there are predecessors $c_{-1}, c_{-2}, \dots, c_{-(n-2)}$ such that

$$x_0 < x_{-n} < c_{-(n-2)} < x_{-(n-1)} < \cdots < c_{-1} < x_{-2} < c_0 < x_{-1}.$$

We consider now the set

$$S = \{y_0 \in R \colon y_n < y_0 < c_{-(n-2)} < y_1 < \cdots < y_{n-3} < c_{-1} < y_{n-2} < c_0 < y_{n-1}\}.$$

The set S is nonempty since $x_{-n} \in S$ and open since f is continuous. Let (a_0, b_0) be the component of S such that $x_{-n} \in (a_0, b_0)$. Since $c_{-(n-2)} \notin S$ and $y_0 \in (a_0, b_0)$ implies $y_0 < c_{-(n-2)}$, we must have $-\infty \leq a_0 < b_0 < c_{-(n-2)}$. We first note that $a_0 > -\infty$. This is so because for every $y_0 \in (a_0, b_0)$ we have $y_n < y_0$ and $y_n \in f^2([c_{-1}, c_0])$, which is a compact set. Thus $y_0 \geq \min f^2([c_{-1}, c_0]) > -\infty$. This implies $a_0 \geq \min f^2([c_{-1}, c_0]) > -\infty$. Since $a_0, b_0 \notin S$ and $y_0 \in (a_0, b_0)$ implies $y_0 \in S$, we conclude by the continuity of f that

$$a_n = a_0 < c_{-(n-2)} < a_1 < \cdots < a_{n-3} < c_{-1} < a_{n-2} < c_0 < a_{n-1}$$

and

$$b_n = b_0 < c_{-(n-2)} < b_1 < \cdots < b_{n-3} < c_{-1} < b_{n-2} < c_0 < b_{n-1}.$$

Hence both a_0 and b_0 are n-periodic points, and since $a_0 < b_0 < b_1 < \cdots < b_{n-1}$, the orbits of a_0 and b_0 are distinct. This completes the proof of the theorem.

COROLLARY 5.3. *If f has an $(n+1)$-periodic loop, $n \geq 3$, then f has two distinct n-periodic loops.*

THEOREM 5.4. $L(\infty) \Rightarrow \cdots \Rightarrow L(4) \Rightarrow L(3) \Rightarrow P(5) \Rightarrow P(7) \Rightarrow \cdots \Rightarrow L^2(\infty)$.

PROOF. If f has an infinite loop satisfying $x_0 < \cdots < x_{-2} < x_{-1}$, the subset $\{x_0, x_{-1}, \dots, x_{-n}\}$, $n \geq 3$, satisfies $x_0 < x_{-n} < \cdots < x_{-2} < x_{-1}$, and is, therefore, a loop of order n. By Theorem 5.3, $L(n)$ holds. Hence $L(\infty) \Rightarrow L(n)$. By Corollary 5.3, $L(n) \Rightarrow L(n-1)$. The implications $L(3) \Rightarrow P(5) \Rightarrow P(7) \Rightarrow \cdots$ follow from Sarkovskii's theorem. Finally, to prove the implication $P(2n+1) \Rightarrow L^2(\infty)$ for every $n \geq 1$, we note that if $C = (x_0, x_1, \dots, x_{2n})$ is a $(2n+1)$-periodic orbit, then $n(C^+) \neq n(C^-)$, so that the hypothesis of either Theorem 5.1 or Theorem

5.2 is satisfied. In the first case, $L^2(\infty)$ holds by Theorem 5.1. In the second case, Theorem 5.2 implies that $L(\infty)$ holds, and hence that $L(3)$ holds. Now for any three-periodic orbit, the hypothesis of Theorem 5.1 holds trivially. Hence $L^2(\infty)$ holds . This completes the proof.

COROLLARY 5.4. $L^{2^k}(\infty) \Rightarrow \cdots \Rightarrow L^{2^k}(5) \Rightarrow L^{2^k}(3) \Rightarrow P(2^k \cdot 5) \Rightarrow P(2^k \cdot 7) \Rightarrow \cdots \Rightarrow L^{2^{k+1}}(\infty)$.

PROOF. This follows from Theorem 5.4 and Lemma 4.3.

PROOF OF THEOREM (SR). Theorem (SR) follows by combining Theorem 5.4, Corollary 5.4, Lemma 4.3, and Sarkovskii's theorem.

6. Examples and remarks. 1. That certain orbit types imply the existence of infinite loops is implicit in Theorem 5.2 and is strikingly illustrated by the example $f(x) = ax(1 - |x|)$. f has the four periodic orbit $(1/2, a/4, 1/2, a/4)$, where $a \approx 4.411138875$ is given by

$$a^{-1} = \frac{1}{2}\left[\left(\frac{1}{2} + \frac{1}{2}\sqrt{59/27}\right)^{1/3} + \left(\frac{1}{2} - \frac{1}{2}\sqrt{59/27}\right)^{1/3}\right].$$

This orbit satisfies the conditions of Theorem 5.2 and hence guarantees the existence of two infinite loops as well as two periodic loops of each period $n \geq 3$ by Theorem (SR). The results of [**5**] ensure for this example merely the existence of a three-periodic orbit. It should be noted that the "type" of the four-periodic orbit in this example is different from that of a four-periodic loop.

2. The results in this paper offer a novel approach to detecting chaos. Most practical methods for detecting chaos rely, either implicitly or explicitly, on the existence of odd periodic orbits [**2**–**5**]. However, Lemmas 4.1 and 4.2 can be used by finding only a few predecessors of a critical point. Lemma 4.2, in particular, is independent of odd periods. A notable illustration is the example $f(x) = x^2 - s$, for which, at

$$s^* = \frac{1}{3}\left(2 + \left(3\sqrt{33} + 17\right)^{1/3} - \left(3\sqrt{33} - 17\right)^{1/3}\right),$$

$L^2(\infty)$ holds, with no odd period $\neq 1$ being present.

3. (a) Block and Coppel introduce in [**6**] the concept of turbulence. By definition a continuous function $f\colon R \to R$ is turbulent if there exist compact intervals J and K such that $J \cup K \subseteq f(J) \cap f(K)$. They then show that f is turbulent if and only if there are points a, b, c such that $f(a) = a = f(b)$, $f(c) = b$, and either $a < c < b$ or $b < c < a$. This characterization of turbulence can be rephrased by virtue of Lemma 4.1 as follows: A continuous function $f\colon R \to R$ is turbulent if and only if f has an infinite loop. We give in 3(c) a proof of greater transparency of this important characterization. If we say that property T holds if f is turbulent, the lemma to be proved reads

LEMMA 6.1. $T \Leftrightarrow L(\infty)$.

(b) We adopt the following convenient notation. For intervals J and K we write $J \leq K$ if $x \leq y$ whenever $x \in J$ and $y \in K$. It follows that $J \leq K$ or $K \leq J$ if and only if $J \cap K$ is at most a singleton. We also recall the following well-known lemma.

LEMMA 6.2. *If $f: R \to R$ is continuous, and J and K are intervals such that K is compact and $f(J) \supset K$, then there is a minimal compact interval $J' \subset J$ such that $f(J') = K$. The interval J' is minimal with respect to the property $f(J') = K$ if no proper subinterval of J' has this property.*

(c) PROOF OF LEMMA 6.1. Let f be turbulent. We assume without loss of generality that $J \leq K$. Then there are minimal compact intervals $J_1 \subset J$ and $J_2 \subset J$ such that $f(J_1) = K$ and $f(J_2) = J$. Since $f(J_1 \cap J_2) \subset f(J_1) \cap f(J_2) = K \cap J$ and $K \cap J$ is at most a singleton, we conclude from the minimality of J_1 and J_2 that $J_1 \cap J_2$ is at most a singleton and hence we have either $J_1 \leq J_2 \leq K$ or $J_2 \leq J_1 \leq K$. In case $J_1 \leq J_2 \leq K$, we conclude the existence of a critical point $c_0 \in K$ and predecessors $c_{-1} \in J_1$ and $c_{-2} \in J_2$ from the respective conditions $f(K) \supset K$, $f(J_1) = K$, and $f(J_2) = J \supset J_1$. From $J_1 \leq J_2 \leq K$ follows $c_{-1} \leq c_{-2} \leq c_0$. We note now that no equality in the last statement can hold, for that would force J_2 to be a singleton which is impossible. Hence f has an infinite loop by Lemma 4.1. The case $J_2 \leq J_1 \leq K$ is similar.

Conversely, if f has an infinite loop, then there is, in particular, a critical point c_0 and predecessors c_{-1} and c_{-2} satisfying $c_0 < c_{-2} < c_{-1}$ (or $c_0 > c_{-2} > c_{-1}$). We let $J = [c_0, c_{-2}]$ and $K = [c_{-2}, c_{-1}]$ and verify that $J \leq K$ and $f(J) \cap f(K) \supset J \cup K$ hold, i.e., f is turbulent. This completes the proof of Lemma 6.1.

(d) The refinement of the Sarkovskii stratification given in [**6**] is an immediate corollary of Theorem (SR) as is visibly revealed by omitting all finite loops.

(e) The ubiquitous quadratic function $f(x) = ax(1-x)$, which is turbulent if and only if $a \geq 4$, suggests that a unimodal function is turbulent if and only if there is an interval $[a, b]$ such that $f(a) \leq a$, $f(b) \leq a$ and $f([a, b]) = [a, b]$. This is indeed easy to prove as is the following lemma that summarizes various observations.

LEMMA 6.3. *The following are equivalent for a continuous function $f: R \to R$:*

(1) *f is turbulent.*
(2) *f has an infinite loop.*
(3) *f has a fixed point c_0 with predecessors c_{-1} and c_{-2} such that either $c_0 < c_{-2} < c_{-1}$ or $c_0 > c_{-2} > c_{-1}$.*
(4) *There are points a, b, c, $a < c < b$, such that either $f(a) \leq a$, $f(b) \leq a$, and $f(c) \geq b$ or $f(a) \geq b$, $f(b) \geq b$, and $f(c) \leq a$.*

We see that the f in 1 is turbulent and that the parameter value s^* in 2 marks the exact value for which f^2 becomes turbulent.

Finally, we remark that for applications the four-point inequality $x_3 < x_2 < x_0 < x_1$ is a simple sufficient condition for turbulence, as exemplified by the example in 2.

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