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A SHORT PROOF OF THE EQUIVALENCE OF KMP AND RNP IN BANACH LATTICES AND PREDUALS OF VON NEUMANN ALGEBRAS

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ABSTRACT. In this note we give a unified approach to the equivalence between the Krein-Milman property and the Radon-Nikodym property in Banach lattices and preduals of von Neumann algebras.

It is by now well known that, for a Banach lattice, the Krein-Milman property (abbreviated KMP) and the Radon-Nikodym property (RNP) are equivalent [**2**]. The proof given in [**2**] combines techniques of [**1**] and [**4**]. On the other hand, as remarked in [**3**], KMP and RNP are also equivalent for preduals of von Neumann algebras. It is our purpose in this note to give a short proof of both results using the main result of [**1**] and an easy argument. We prove a more general result, in terms of ordered Banach spaces, enhancing both results above. Let E be an ordered Banach space [**6**] with a positive cone E_+. Then the bidual E'' of E is also an ordered Banach space whose positive cone will be denoted by E''_+. We identify E with the canonical image of E into E''. We say that E_+ is solid in E''_+ if, for every $z \in E''_+$ and every $x \in E$ such that $z \leq x$, we have that $z \in E_+$. We can now state our theorem.

THEOREM. *Let E be an ordered Banach space such that E_+ is solid in E''_+. Let C be a closed, bounded, convex subset of E_+ with the KMP. Then C has the RNP.*

PROOF. If C does not have the RNP, there exists a closed, convex subset D of C such that D does not contain any extreme point of its w^*-closure $\tilde{D}$ in E''. Let $\mathrm{ex}(D)$, $\mathrm{ex}(\tilde{D})$ denote the set of extreme points of D and $\tilde{D}$, respectively. Let $z \in \mathrm{ex}(D)$. Since $z \notin \mathrm{ex}(\tilde{D})$, there exists $a, b > 0$, $a + b = 1$, and $z_1, z_2 \in \tilde{D}$ such that $z = az_1 + bz_2$. If $z_1 \in E$, then $bz_2 \in E$. Since $b > 0$, $z_2 \in E$. Hence, $z_1, z_2 \in \tilde{D} \cap E = D$ and $z \notin \mathrm{ex}(D)$. This contradiction proves that $z_1 \notin E$. Similarly, $z_2 \notin E$. But $\tilde{D} \subseteq E''_+$, $z_1 \leq a^{-1}z$, $z_2 \leq b^{-1}z$ together with our assumption imply that $z_1, z_2 \in E$. This contradiction shows that $\mathrm{ex}(D) = \varnothing$ and C does not have the KMP.

A Banach lattice E with the KMP does not contain a copy of c_0. Hence E is weakly sequentially complete and E_+ is solid in E''_+ [**5**, II.5]. Taking $C := \{x \in E_+ : \|x\| \leq 1\}$, Theorem 1 proves that C is a Radon-Nikodym set. Therefore, positive linear operators from $L^1[0,1]$ into E are representable. Since every

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bounded, linear operator $T\colon L^1[0,1] \to E$ is the difference of two positive operators [**5**, IV.1.5], we obtain

COROLLARY 1 [**2**]. *If E is a Banach lattice with the KMP, then E has the RNP.*

Let M_* be the predual of a von Neumann algebra M. M_* is an ordered Banach space whose positive cone $(M_*)_+$ (= the set of all normal, positive linear functionals on M [**7**, III.4] is solid in $(M')_+$ [**7**, III.2.14]). If M_* has the KMP, by Theorem 1, $C\colon = \{f \in (M_*)_+ : \|f\| \leq 1\}$ has the RNP. Using a compactness argument and the fact that there exists a contractive projection P from M' onto M_* with $P((M')_+) \subseteq (M_*)_+$, one proves that all bounded, linear operators $T\colon L^1[0,1] \to M$ with T (unit ball of L^1) $\subseteq C - C$ can be written as $T = T_1 - T_2$ where $T_1, T_2\colon L^1[0,1] \to M$ map the unit ball of $L^1[0,1]$ into C, hence are representable. Therefore, the real part of M (= the set of normal hermitian functionals) has the RNP. Since any functional $f \in M_*$ can be written as $f = f_1 + if_2$ where $f_1 = 2^{-1}(f + f^*)$, $f_2 = (2i)^{-1}(f - f^*)$, and $f^*(x) := \overline{f(x^*)}$, $x \in M$, it is a routine argument to show that M_* has the RNP. We have obtained

COROLLARY 2 [**3**]. *In the predual of a von Neuman algebra the KMP and the RNP are equivalent.*

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