A SHORT PROOF OF THE EQUIVALENCE
OF KMP AND RNP IN BANACH LATTICES AND PREDUALS
OF VON NEUMANN ALGEBRAS

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ABSTRACT. In this note we give a unified approach to the equivalence between the Krein-Milman property and the Radon-Nikodym property in Banach lattices and preduals of von Neumann algebras.

It is by now well known that, for a Banach lattice, the Krein-Milman property (abbreviated KMP) and the Radon-Nikodym property (RNP) are equivalent [2]. The proof given in [2] combines techniques of [1] and [4]. On the other hand, as remarked in [3], KMP and RNP are also equivalent for preduals of von Neumann algebras. It is our purpose in this note to give a short proof of both results using the main result of [1] and an easy argument. We prove a more general result, in terms of ordered Banach spaces, enhancing both results above. Let $E$ be an ordered Banach space [6] with a positive cone $E_+$. Then the bidual $E''$ of $E$ is also an ordered Banach space whose positive cone will be denoted by $E''_+$. We identify $E$ with the canonical image of $E$ into $E''$. We say that $E_+$ is solid in $E_+'$ if, for every $z \in E_+'$ and every $x \in E$ such that $z \leq x$, we have that $z \in E_+$. We can now state our theorem.

**THEOREM.** Let $E$ be an ordered Banach space such that $E_+$ is solid in $E_+'$. Let $C$ be a closed, bounded, convex subset of $E_+$ with the KMP. Then $C$ has the RNP.

**PROOF.** If $C$ does not have the RNP, there exists a closed, convex subset $D$ of $C$ such that $D$ does not contain any extreme point of its $w^*$-closure $\bar{D}$ in $E''$. Let $\text{ex}(D)$, $\text{ex}(\bar{D})$ denote the set of extreme points of $D$ and $\bar{D}$, respectively. Let $z \in \text{ex}(D)$. Since $z \notin \text{ex}(\bar{D})$, there exists $a, b > 0$, $a + b = 1$, and $z_1, z_2 \in D$ such that $z = az_1 + bz_2$. If $z_1 \in E$, then $bz_2 \in E$. Since $b > 0$, $z_2 \in E$. Hence, $z_1, z_2 \in \bar{D} \cap E = D$ and $z \notin \text{ex}(D)$. This contradiction proves that $z_1 \notin E$. Similarly, $z_2 \notin E$. But $\bar{D} \subseteq E'_+$, $z_1 \leq a^{-1}z$, $z_2 \leq b^{-1}z$ together with our assumption imply that $z_1, z_2 \in E$. This contradiction shows that $\text{ex}(D) = \emptyset$ and $C$ does not have the KMP.

A Banach lattice $E$ with the KMP does not contain a copy of $c_0$. Hence $E$ is weakly sequentially complete and $E_+$ is solid in $E''_+$ [5, II.5]. Taking $C := \{x \in E_+ : \|x\| \leq 1\}$, Theorem 1 proves that $C$ is a Radon-Nikodym set. Therefore, positive linear operators from $L^1[0,1]$ into $E$ are representable. Since every...
bounded, linear operator \( T : L^1[0,1] \to E \) is the difference of two positive operators [5, IV.1.5], we obtain

**COROLLARY 1 [2]**. If \( E \) is a Banach lattice with the KMP, then \( E \) has the RNP.

Let \( M_* \) be the predual of a von Neumann algebra \( M \). \( M_* \) is an ordered Banach space whose positive cone \( (M_*)_+ \) (= the set of all normal, positive linear functionals on \( M \) [7, III.4] is solid in \( (M')_+ \) [7, III.2.14]). If \( M_* \) has the KMP, by Theorem 1, \( C := \{ f \in (M_*)_+ : \|f\| \leq 1 \} \) has the RNP. Using a compactness argument and the fact that there exists a contractive projection \( P \) from \( M' \) onto \( M_* \) with \( P((M')_+) \subseteq (M_*)_+ \), one proves that all bounded, linear operators \( T : L^1[0,1] \to M \) with \( T \) (unit ball of \( L^1 \)) \subseteq \( C \), map the unit ball of \( L^1[0,1] \) into \( C \), hence are representable. Therefore, the real part of \( M \) (= the set of normal hermitian functionals) has the RNP. Since any functional \( f \in M_* \) can be written as \( f = f_1 + i f_2 \) where \( f_1 = 2^{-1}(f + f^*) \), \( f_2 = (2i)^{-1}(f - f^*) \), and \( f^*(x) := \overline{f(x)} \), \( x \in M \), it is a routine argument to show that \( M_* \) has the RNP. We have obtained

**COROLLARY 2 [3]**. In the predual of a von Neumán algebra the KMP and the RNP are equivalent.

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