

**A UNIQUENESS RESULT FOR VISCOSITY SOLUTIONS  
OF SECOND ORDER FULLY NONLINEAR  
PARTIAL DIFFERENTIAL EQUATIONS**

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**ABSTRACT.** In this note we extend some recent results of R. Jensen concerning the uniqueness of viscosity solutions of scalar, second order, fully nonlinear, elliptic, possibly degenerate, partial differential equations.

In this note we consider the problem of uniqueness of viscosity solutions of scalar second order nonlinear elliptic, possibly degenerate, partial differential equations of the form

$$(1) \quad F(D^2u, Du, u, x) = 0 \quad \text{in } \Omega.$$

Here  $\Omega$  is an open subset of  $\mathbf{R}^N$ ,  $Du = (u_{x_1}, \dots, u_{x_N})$ ,  $D^2u = (\partial u / \partial x_i \partial x_j)_{1 \leq i, j \leq N}$ , and  $F: S^{N \times N} \times \mathbf{R}^N \times \mathbf{R} \times \Omega \rightarrow \mathbf{R}$  is a continuous function of its arguments, where  $S^{N \times N}$  is the space of symmetric  $N \times N$  matrices. We also assume that equation (1) is elliptic, possibly degenerate, i.e. we assume that for all  $A, B \in S^{N \times N}$ ,  $p \in \mathbf{R}^N$ ,  $s \in \mathbf{R}$  and  $x \in \Omega$  we have

$$(2) \quad F(A, p, s, x) \leq F(B, p, s, x) \quad \text{if } A \geq B,$$

where  $A \geq B$  means  $(Ay, y) \geq (By, y)$  for all  $y \in \mathbf{R}^N$ . Equations of the type (1) arise naturally in the theory of optimal stochastic control (cf. W. H. Fleming and R. W. Rishel [5], P.-L. Lions [10, 11]) and stochastic differential games (cf. R. Jensen and P.-L. Lions [8], W. H. Fleming and P. E. Souganidis [6]).

One of the many equivalent formulations of the notion of viscosity solution of (1) is the following.

**DEFINITION.** A continuous function  $u$  is a viscosity subsolution (resp. supersolution) of (1) if

$$(3) \quad F(D^2\phi(x_0), D\phi(x_0), u(x_0), x_0) \leq 0$$

(resp.

$$(4) \quad F(D^2\phi(x_0), D\phi(x_0), u(x_0), x_0) \geq 0)$$

for all  $\phi \in C^2(\Omega)$  and  $x_0$  local maximum (resp. local minimum) of  $u - \phi$ . The function  $v$  is a viscosity solution of (1), if it is both sub- and supersolution.

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M. G. Crandall and P.-L. Lions [4] introduced the notion of viscosity solutions and proved general uniqueness results about first order problems of the form (1) (a new presentation of these uniqueness proofs may be found in M. G. Crandall, H. Ishii and P.-L. Lions [3]). In the case where  $F$  is convex or concave with respect to  $(D^2u, Du)$ , i.e., stochastic control case, P.-L. Lions [10, 11] proved that viscosity solutions of (1) depend continuously on their boundary data. More recently, R. Jensen [7] proved that, in the general case where  $F$  is neither convex nor concave (stochastic differential games case), Lipschitz continuous viscosity solutions of (1) are unique. Although he assumed that  $F$  is independent of  $x$ , a careful examination of the proofs reveals that some dependence on  $x$  is allowable. In this note we relax the Lipschitz continuity assumption and address the uniqueness question for  $F$ 's which have some kind of  $x$  dependence; for example

$$(5) \quad F(A, p, s, x) = G(A) + H(p, s, x).$$

We next formulate our main result.

**THEOREM.** *Let  $u, v \in \text{BUC}(\bar{\Omega})$  be respectively viscosity subsolution and supersolution of (1). We assume that, in addition to (2),  $F$  also satisfies*

$$(6) \quad F \text{ is uniformly continuous with respect to } p, \text{ uniformly for } A \in S^{N \times N}, x \in \Omega, \text{ and } p \text{ and } t \text{ bounded.}$$

$$(7) \quad \text{For every } 0 < R \text{ there exists a } \gamma_R > 0 \text{ such that } F(A, p, t, x) - F(A, p, s, x) \geq \gamma_R(t - s) \text{ for all } R \geq t \geq s \geq -R, A, p \text{ and } x.$$

$$(8) \quad \text{For every } 0 < R \text{ there exists a } \omega_R: \mathbf{R} \rightarrow \mathbf{R} \text{ such that } \omega_R(s) \rightarrow 0 \text{ as } s \downarrow 0 \text{ and } F(A, \lambda(x - y), t, x) - F(A, \lambda(x - y), t, y) \geq -\omega_R(\lambda|x - y|^2 + |x - y|) \text{ for all } A, \lambda \geq 1, x, y \text{ and } |t| \leq R.$$

and, if  $\Omega$  is unbounded,

$$(9) \quad F \text{ is uniformly continuous with respect to } (A, p) \text{ uniformly for } A, x, t \text{ and } p \text{ bounded.}$$

Then

$$(10) \quad \sup_{\bar{\Omega}}(u - v)^+ = \sup_{\partial\Omega}(u - v)^+.$$

If  $\Omega = \mathbf{R}^N$ , this means that  $u \leq v$  in  $\mathbf{R}^N$ .

**REMARK.** Assumption (8) is satisfied by functions  $F$  like (5) but unfortunately does not include linear  $F$ 's with variable coefficients.

The proof of the theorem follows closely the proof of the main result of [7]. It consists of two steps. The first step says that (10) holds, if  $u$  and  $v$  are sufficiently smooth (but not classical) viscosity sub- and supersolutions of (1). It is based on a variant of the generalized maximum principle given in R. Jensen [7] (cf. Alexandrov [1], J. M. Bony [2], P.-L. Lions [12] for previous related results). The second step consists of establishing that given  $u$  and  $v$  as in the statement of the theorem, we can find approximations which satisfy the assumptions of the first step. The approximations used here are due to J. M. Lasry and P.-L. Lions [8] and are different to the ones used in [7]. (For a comparison see at the end of the note.) They allow us to extend the results of [7] in the sense that solutions are only continuous functions. Next we formulate these two steps in the form of two propositions. The first proposition is taken directly from [7], so we omit its proof.

PROPOSITION 1. *Let  $u$  (resp.  $v$ ) be a bounded Lipschitz continuous semiconvex (resp. semiconcave) viscosity subsolution (resp. supersolution) of (1). If  $F$  satisfies (6)–(8) and (9), if  $\Omega$  is unbounded, then (10) holds.*

PROPOSITION 2. *Let  $u, v \in \text{BUC}(\bar{\Omega})$  be respectively viscosity subsolution and supersolution of (1). If  $F$  is nondecreasing with respect to  $t$  and also satisfies (8) then, for each  $\varepsilon > 0$ , there exist bounded Lipschitz continuous functions  $u_\varepsilon$  and  $v^\varepsilon$  on  $\bar{\Omega}_\varepsilon = \{x \in \Omega: d^2(x, \partial\Omega) \geq \varepsilon(4R_0 + 1)\}$  ( $R_0 > \max(\|u\|_\infty, \|v\|_\infty)$ ) such that  $u_\varepsilon$  is semiconvex on  $\Omega_\varepsilon$ ,  $v^\varepsilon$  is semiconcave on  $\Omega_\varepsilon$  and  $u_\varepsilon, v^\varepsilon$  are respectively viscosity subsolution and supersolution of*

$$(11) \quad F(D^2u_\varepsilon, Du_\varepsilon, u_\varepsilon, x) = k(\varepsilon) \quad \text{in } \Omega_\varepsilon$$

and

$$(12) \quad F(D^2v^\varepsilon, Dv^\varepsilon, v^\varepsilon, x) = -k(\varepsilon) \quad \text{in } \Omega_\varepsilon$$

where  $k(\varepsilon) \rightarrow 0$  as  $\varepsilon \downarrow 0$ . Finally,

$$(13) \quad \sup_{\bar{\Omega}_\varepsilon} |u_\varepsilon - u|, \sup_{\bar{\Omega}_\varepsilon} |v^\varepsilon - v| \rightarrow 0 \quad \text{as } \varepsilon \downarrow 0.$$

We briefly sketch the proof of Proposition 2.

PROOF OF PROPOSITION 2. We define  $u_\varepsilon$  and  $v^\varepsilon$  by

$$(14) \quad u_\varepsilon(x) = \sup_{\bar{\Omega}} \left\{ u(y) - \frac{1}{2\varepsilon} |x - y|^2 \right\}$$

and

$$(15) \quad v^\varepsilon(x) = \inf_{\bar{\Omega}} \left\{ v(y) + \frac{1}{2\varepsilon} |x - y|^2 \right\}.$$

It follows that  $|u_\varepsilon|, |v^\varepsilon| \leq R_0$ ,  $u_\varepsilon \geq u$  on  $\Omega$ ,  $v^\varepsilon \leq v$  on  $\Omega$ ,  $u_\varepsilon$  and  $v^\varepsilon$  are respectively semiconvex and semiconcave; more precisely,  $u_\varepsilon(x) + |x|^2/2\varepsilon$  is convex in  $\Omega$  while  $v^\varepsilon(x) - |x|^2/2\varepsilon$  is concave in  $\Omega$ . Finally, if  $m$  is a modulus of continuity of  $u$  and  $v$ , the sup and inf in (14) and (15) respectively may be restricted to  $y$ 's such that  $|x - y|^2 \leq 2\varepsilon m(|x - y|)$  and  $|x - y|^2 \leq 4\varepsilon R_0$ . So, if  $x \in \Omega_\varepsilon$ , then the inf and sup are attained at points  $y \in \Omega$ . Next we show how one obtains (11). Since (12) is proved in exactly the same way we omit its proof. To this end, suppose that  $u_\varepsilon - \phi$  has a local maximum at  $x_0 \in \Omega_\varepsilon$ , where  $\phi \in C^2(\Omega_\varepsilon)$ . Let  $y_0 \in \bar{\Omega}$  be a maximum point of the sup in (14). Since  $x_0 \in \Omega_\varepsilon$ , we have that  $y_0 \in \Omega$ . Moreover, we have the inequality

$$(16) \quad u(y_0) - \frac{1}{2\varepsilon} |x_0 - y_0|^2 - \phi(x_0) \geq u(y) - \frac{1}{2\varepsilon} |x - y|^2 - \phi(y)$$

for  $x$  and  $y$  near  $x_0$  and  $y_0$  respectively. This yields that  $D\phi(x_0) = (y_0 - x_0)/\varepsilon$  and that  $y \rightarrow u(y) - \phi(x_0 - y_0 + y)$  admits a maximum at  $y_0$ . Since  $u$  is a viscosity subsolution of (1) we have

$$F(D^2\phi(x_0), D\phi(x_0), u(y_0), y_0) \leq 0.$$

Applying now (8) and using the monotonicity of  $F$  with respect to  $t$ , we may replace  $(u(y_0), y_0)$  above by  $(u_\varepsilon(x_0), x_0)$  by making an error  $k(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . The proof is complete.

We now compare the approximations  $u_\varepsilon$  and  $v^\varepsilon$  above to the ones obtained in [7]. Roughly speaking, the approximations in [7] are obtained by solving the eikonal equation  $|DU| = 1$  over or below the graph of  $u$  and  $v$  and then taking the level set  $U = \varepsilon$  as the graph of the  $\varepsilon$  approximation; this requires  $u$  and  $v$  to be Lipschitz continuous. Here we solve the time dependent eikonal equations

$$\frac{\partial w}{\partial t} + \frac{1}{2}|\nabla w|^2 = 0 \quad \text{or} \quad \frac{\partial w}{\partial t} - \frac{1}{2}|\nabla w|^2 = 0$$

with initial data  $w(\cdot, 0) = u$  or  $v$  and use  $w(\cdot, \varepsilon)$  as our approximations. This can be done with  $u$  and  $v$  only continuous.

We conclude by remarking that all the above apply without any difficulty to time-dependent problems. Since the results are almost identical, we do not give any precise statement here.

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