PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 102, Number 4, April 1988

LARGE ROOTS YIELD LARGE COEFFICIENTS: AN ADDENDUM TO 'THE ROOTS OF A POLYNOMIAL VARY CONTINUOUSLY AS A FUNCTION OF THE COEFFICIENTS'

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(Communicated by Dennis Burke)

ABSTRACT. We provide the details to the argument that the map $\hat{\sigma}$, defined in the above named paper, is open. We do this by including two trivial arguments that a polynomial with a big root must also have a big coefficient.

This note is presented as a supplement to the article *The roots of a polynomial vary continuously as a function of the coefficients*, [**2**]. It is the response to numerous queries concerning the argument that the map $\hat{\sigma}\colon \mathbf{C}^n/\sim \longrightarrow \mathbf{C}^n$ is open with respect to the topology induced on $\mathbf{C}^n$ by the euclidean norm.

We recall the setup of the aforementioned article. A polynomial $P(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ is identified with the vector $(a_1, \ldots, a_n)$ in $\mathbf{C}^n$. For such a polynomial we let $\{\xi_1, \ldots, \xi_n\}$ denote the sequence of roots and recall for each $1 \leq j \leq n$,

$$a_j = \sigma_j(\xi_1, \ldots, \xi_n)$$

for symmetric polynomials $\sigma_1, \ldots, \sigma_n$ in n variables. Let $\sigma\colon \mathbf{C}^n \to \mathbf{C}^n$ be defined by

$$\sigma(\xi) \doteq (\sigma_1(\xi), \ldots, \sigma_n(\xi)).$$

We are using the notation $\xi \doteq (\xi_1, \ldots, \xi_n)$. The map $\hat{\sigma}$ is the map naturally induced by σ on the quotient space $\mathbf{C}^n/\sim$, the relation "$\sim$" being defined on $\mathbf{C}^n$ by setting equal any two vectors for which the ordered component sequence of one is a permutation of the ordered component sequence of the other.

It follows immediately from the arguments in the previous paper [**2**, p. 391], that an element, ξ_0, in the interior of a ball $B(0, M_0) \doteq \{\xi \in \mathbf{C}^n/\sim\colon d(0,\xi) < M_0\}$ must have its image, $\hat{\sigma}(\xi_0)$, in the relative interior of the compact set $\hat{\sigma}(\overline{B(0, M_0)})$. The only way $\hat{\sigma}(\xi_0)$ could fail to be in the interior of $\hat{\sigma}(\overline{B(0, M_0)})$, is for $\hat{\sigma}(\xi_0)$ to belong to the topological boundary of $\hat{\sigma}(\overline{B, (0, M_0)})$. It is clear [**2**, p. 391] this would imply $\hat{\sigma}(\xi_0)$ belongs to the topological boundary of $\hat{\sigma}(\overline{B(0, M)})$ for all $M > M_0$. Hence there would exist a sequence $\{\xi_n\}_{n=1}^{\infty} \subset \mathbf{C}^n/\sim$ such that $d(0, \xi_n) \to \infty$ while $|\hat{\sigma}(\xi_n)| \to |\hat{\sigma}(\xi_0)| < \infty$, which contradicts the fact that a polynomial with a large root must have a large coefficient.

The easiest way to observe that large roots yield large coefficients is as follows. Suppose $|\xi| > 1$ and $P(\xi) = 0$. So $-\xi^n = a_1\xi^{n-1} + \cdots + a_n$. Hence $|\xi|^n \leq |a_1||\xi|^{n-1} + \cdots + |a_n|$. Dividing this inequality by $|\xi|^{n-1}$, and observing that $|\xi| > 1$ implies $|\xi| < |a_1| + \cdots + |a_n|$. Thus $|a_j| > \frac{1}{n}|\xi|$ for some $1 \leq j \leq n$.

Received by the editors September 9, 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 26C10, 30C15.

A somewhat more interesting result along this line is the following

THEOREM 1. *Suppose P is the polynomial given by $P(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ and $P(\xi) = 0$. Let $|\xi| = M$ and suppose M is so large that $M^{n-1} - M^{n-2} - \cdots - M > 1$. Then there exist j such that $a_j > M/2$.*

PROOF. Suppose for $1 \leq j \leq n - 1$, $|a_j| < M/2$. We claim this implies $|a_n| > M/2$. Since ξ is a root of P, there are numbers $b_1, \ldots, b_{n-1}$ such that

$$\begin{aligned} P(z) &= (z - \xi)(z^{n-1} + b_1 z^{n-2} + \cdots + b_{n-1}) \\ &= z^n + (b_1 - \xi) z^{n-1} + (b_2 - \xi b_1) z^{n-2} + \cdots + (-\xi b_{n-1}). \end{aligned}$$

It follows that

$$\begin{aligned} a_1 &= b_1 - \xi, \\ a_j &= b_j - \xi b_{j-1} \quad \text{if } 2 \leq j \leq n-1, \text{ and} \\ a_n &= -\xi b_{n-1}. \end{aligned}$$

The assumption, $|a_j| < M/2$ for $1 \leq j \leq n - 1$ now implies

$$\begin{gathered} |b_1| > |\xi| - M/2 = M/2, \\ |b_2| > |\xi||b_1| - M/2 > (M^2 - M)/2; \quad \text{etc.} \end{gathered}$$

Continuing, we obtain

$$|b_{n-1}| > (M^{n-1} - M^{n-2} - \cdots - M)/2.$$

The assumption on M implies $|b_{n-1}| > 1/2$. Thus $|a_n| = |\xi||b_{n-1}| > M/2$. This completes the argument, for it shows that either $|a_n| > M/2$ or $|a_j| > M/2$ for some $1 \leq j \leq n - 1$.

One final comment is in order. We are grateful to the authors of an earlier paper entitled *The space of unordered tuples of complex numbers* [**1**], for providing to us a copy of their paper which contains essentially the same material.

BIBLIOGRAPHY

1. Rajendra Bhatia and Kalyan Mukherjea, *The space of unordered tuples of complex numbers*, Linear Algebra Appl. **52/53** (1983), 765–768.
2. Gary Harris and Clyde Martin, *The roots of a polynomial vary continuously as a function of the coefficients*, Proc. Amer. Math. Soc. **100** (1987), 390–392.

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