4-PLANAR GEODESIC KAEBLER IMMERSIONS INTO A COMPLEX PROJECTIVE SPACE

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ABSTRACT. If \( f \) is a proper 4-planar geodesic Kaehler immersion of a connected complete Kaehler manifold \( M^n (n \geq 2) \) into \( CP^n(c) \), then \( M^n = CP^n(c/4) \) and \( f \) is equivalent to the 4th Veronese map.

0. Introduction. Let \( \overline{M} \) be a Riemannian manifold. A curve \( \tau: I \rightarrow \overline{M} \) defined on an open interval \( I \) is said to be \( d \)-planar if there exist an open interval \( I_s (s \in I_s \subset I) \) and a \( d \)-dimensional totally geodesic submanifold \( P_s \) for each \( s \in I \) such that \( \tau(I_s) \subset P_s \). Moreover, a \( d \)-planar curve \( \tau \) is said to be proper if it is not \((d-1)\)-planar on each open subinterval of \( I \). An isometric immersion \( f: M \rightarrow \overline{M} \) of a Riemannian manifold \( M \) is called a (resp. proper) \( d \)-planar geodesic immersion if \( \tau = f \circ \gamma \) is (resp. proper) \( d \)-planar geodesic for every geodesic \( \gamma: I \rightarrow M \). 1-planar geodesic immersions are totally geodesic. 2-planar geodesic immersions into real space forms were classified in [7] (for other treatment, see [1]).

When the ambient manifold is a complex projective space \( CP^n(c) \) with constant holomorphic sectional curvature \( c \), 2-planar and odd order proper planar geodesic Kaehler immersions were classified in [5 and 6], respectively. In this paper, we shall study proper 4-planar geodesic Kaehler immersions into \( CP^n(c) \).

1. Notation and basic equations (cf. [2]). For a Kaehler immersion \( f: M \rightarrow CP^n(c) \), the second fundamental form and Weingarten map corresponding to a normal vector field \( \xi \) will be denoted by \( H \) and \( A_\xi \), respectively. Gauss and Weingarten’s equations are given by

\[
\nabla_X Y = \nabla_X Y + H(X,Y), \quad \nabla_X \xi = -A_\xi X + \nabla_X^L \xi
\]

for all tangent vector fields \( X \) and \( Y \) on \( M \), where \( \nabla, \nabla, \) and \( \nabla^L \) denote the covariant differentiation of \( \overline{M}, M, \) and the normal bundle, respectively. Let \( R \) be the curvature tensor and \( J \) the complex structure of \( M \). The structure equation of Gauss is given by

\[
R(X,Y)Z = (c/4)\{(Y,Z)X - (X,Z)Y + (JY,Z)JX - (JX,Z)JY - 2(JX,Y)JZ\} + A_H(Y,Z)X - A_H(X,Z)Y.
\]
The structure equation of Codazzi reduces to \((DH)(X,Y,Z) = (DH)(Y,X,Z)\), where
\[
(DH)(X,Y,Z) = \nabla^2_X H(Y,Z) - H(\nabla_X Y, Z) - H(Y, \nabla_X Z).
\]
Let \(\overline{J}\) be the complex structure of \(CP^m(c)\). Since \(\overline{\nabla J} = 0\), we have
\[
(1.3) \quad H(JX,Y) = JH(X,Y).
\]
If \(H\) satisfies \(\|H(X,X)\|^2 = \lambda^2(x)\) for all unit vectors \(X \in T_xM\) and each \(x \in M\), then the immersion \(f\) is said to be isotropic (or \(\lambda\)-isotropic). We note that \(f\) is isotropic if and only if \((H(X,X), H(Y,Y)) = 0\) for any orthonormal vectors \(X\) and \(Y\) at every point.

2. Proper 4-planar geodesic Kaehler immersions. Let \(M\) be a connected complete Riemannian manifold and \(f: M \rightarrow CP^m(c)\) a proper \(d\)-planar geodesic immersion. We first prove

**Lemma 2.1.** For each geodesic \(\gamma\) of \(M\), there exists a unique \(d\)-dimensional totally geodesic submanifold \(P_\gamma\) such that \(\tau((-\infty, \infty)) \subset P_\gamma\), where \(\tau = f \circ \gamma\). Each \(P_\gamma\) is complex or totally real.

**Proof.** Let \(u \in (-\infty, \infty)\) be arbitrarily fixed and put \(P_\gamma = P_u\). If we define a set \(U = \{s \in (-\infty, \infty): \tau(s) \in P_\gamma\}\), then \(U\) is nonempty and closed. Let \(v \in U\).
Consider a finite cover \(\{I_1 = I_u, I_2, \ldots, I_k = I_v\}\) of \([u, v]\) where we have assumed \(u < v\) without loss of generality. Noting that \(f\) is proper \(d\)-planar geodesic and the intersection of two totally geodesic submanifolds is a totally geodesic submanifold, we see that \(P_\gamma = P_u = P_{s_1} = \cdots = P_v\). Therefore, \(I_v \subset U\), i.e., \(U\) is open and hence \(U = (-\infty, \infty)\). The uniqueness of \(P_\gamma\) is easily derived from the assumption that \(f\) is proper \(d\)-planar geodesic. It is well known that a submanifold in \(CP^m(c)\) is complex or totally real if and only if the second fundamental form \(H\) of the submanifold satisfies \((DH)(X,Y,Z) = (DH)(Y,X,Z)\) (cf. [6, (1.9), p. 300]). Therefore, since \(P_\gamma\) is totally geodesic, we have the assertion. Q.E.D.

Let \(x \in M\), \(X \in U_xM\) (unit tangent sphere at \(x\)), and let \(\gamma\) be the unit speed geodesic such that \(\gamma(0) = x\) and \(\dot{\gamma}(0) = X\). Then \(\tau = f \circ \gamma\) satisfies
\[
\dot{\tau}(0) = f_* X,
\]
\[
\nabla^1_X \dot{\tau} = H(X, X),
\]
\[
\nabla^2_X \dot{\tau} = -A_{H(X,X)} X + (DH)(X,X,X).
\]
Higher order covariant derivatives of \(\dot{\tau}\) in the direction \(X\) can be also obtained by using Gauss and Weingarten equations (1.1). Note that all covariant derivatives of \(\dot{\tau}\) are tangent to \(P_\gamma\). Define a function \(\phi\) on the unit tangent sphere bundle \(UM\) over \(M\) by
\[
\phi(X) = \det((\nabla^i_X \dot{\tau}, \nabla^j_X \dot{\tau})_{i,j=0,1,\ldots,d-1})
\]
\[
= \text{Gramian of vectors } X, \nabla^1_X \dot{\tau}, \ldots, \nabla^{d-1}_X \dot{\tau}\n\]
for \(X \in UM\). If \(\phi(X) \neq 0\), then vectors \(X, \nabla^1_X \dot{\tau}, \ldots, \nabla^{d-1}_X \dot{\tau}\) form a base of \(T_xP_\gamma\).
**Lemma 2.2.** Let $S$ be any connected component of the set \( \{ X \in UM : \phi(X) \neq 0 \} \). Then \( P_{\gamma X} \) is complex for every \( X \in S \) or totally real for every \( X \in S \) where \( \gamma^X \) denotes the geodesic tangent to \( X \).

**Proof.** Assume that there exist \( X \) and \( Y \) in \( S \) such that \( P_{\gamma X} \) is complex and \( P_{\gamma Y} \) is totally real. Since \( S \) is arcwise connected, there is a smooth curve \( X(t) \) in \( S \) such that \( X(0) = X \) and \( X(1) = Y \). Consider a function \( \psi \) on \([0,1]\) defined by

\[
\psi(t) = \text{Sup}\{ \langle JX(t), Z \rangle : Z \in T_{\pi(X(t))} P_{\gamma X(t)}, \| Z \| = 1 \},
\]

where \( \pi : UM \to M \) is the projection. Since

\[
T_{\pi(X(t))} P_{\gamma X(t)} = \text{Span}\{ X(t), \nabla X(t) \hat{t}, \ldots, \nabla^{d-1} X(t) \hat{t} \} \quad (\tau_t = f \circ \gamma^X(t))
\]

which is a smooth curve in the Grassmann bundle of \( d \)-planes over \( M \), we see that \( \psi \) is a continuous function. Moreover, \( P_{\gamma X(t)} \) is complex or totally real (Lemma 2.1) and hence \( \psi(t) = 1 \) or \( 0 \) for each \( t \in [0,1] \). Thus \( \psi \) is constant. However \( \psi(0) = 1 \) and \( \psi(1) = 0 \). Q.E.D.

Now we explain Kaehler immersions into \( CP^m(c) \) of symmetric Kaehler manifolds of compact type. Let \( M \) be an irreducible symmetric Kaehler manifold of compact type and \( k \) a positive integer. In [4], Nakagawa and Takagi constructed a full equivariant Kaehler imbedding \( f_k : M \to CP^m(c) \) which is called the \( k \)th full Kaehler imbedding of \( M \). Moreover, in [8] Takagi and Takeuchi constructed a full Kaehler imbedding of a (not necessarily irreducible) symmetric Kaehler manifold \( M \) of compact type into \( CP^m(c) \) as follows. Let \( M_i (i = 1, \ldots, q) \) be the irreducible components of \( M \), i.e., \( M = M_1 \times \cdots \times M_q \) and \( f_k : M_i \to CP^m(c) \) be the \( k \)th full Kaehler imbedding of \( M_i \). Define a full Kaehler imbedding \( S_q : CP^m_1(c) \times \cdots \times CP^m_q(c) \to CP^m(c) \) by the multifold tensor product of the homogeneous coordinates where \( m = (m_1 + 1) \times \cdots \times (m_q + 1) - 1 \) and we notice that \( S_2 \) is the Segre imbedding. Then \( S_q \circ (f_{k_1} \times \cdots \times f_{k_q}) \) becomes a full equivariant Kaehler imbedding of \( M \) into \( CP^m(c) \). In [4 and 9], it was shown that any full Kaehler immersion of a compact symmetric Kaehler manifold into \( CP^m(c) \) is obtained in this way. In particular, if \( M = CP^n(c/k) \), then the \( k \)th full Kaehler imbedding \( V_k^n : CP^n(c/k) \to CP^m(k)(c) \) is called the \( k \)th Veronese map which is defined by

\[
[z_i]_{0 \leq i \leq n} \mapsto \left( \frac{k!}{k_0! \cdots k_n!} \right)^{1/2} \left[ z_0^{k_0} \cdots z_n^{k_n} \right]_{k_0 + \cdots + k_n = k},
\]

where \([*]\) means the point of the projective space with the homogeneous coordinate \(*\) and \( m(k) = (n + k) - 1 \).

The following two lemmas were proved in [6].

**Lemma 2.3 (cf. the proof of Proposition 2.1 in [6]).** Let \( f : M \to CP^m(c) \) be a Kaehler immersion of a connected complete Kaehler manifold \( M \) into \( CP^m(c) \). Assume that \( \langle H(X,X), (DH)(X,X,X) \rangle = 0 \) for every \( X \in TM \). Then \( M \) is a compact simply connected symmetric Kaehler manifold.

**Lemma 2.4 (cf. the proof of Theorem 2.3 in [6]).** Let \( f : M \to CP^m(c) \) be a proper \( d \)-planar geodesic Kaehler immersion of a symmetric Kaehler
manifold of compact type. Then $M^n = CP^n(c/d)$ and $f$ is equivalent to $i \circ V^n_d$, where $i:CP^m(d)(c) \to CP^m(c)$ is a totally geodesic imbedding.

Here we note that the equivalence of two isometric immersions $f$ and $f'$ of a Riemannian manifold into a Riemannian manifold $\bar{M}$ is defined as follows: If there exists an isometry $F$ of $\bar{M}$ such that $f' = F \circ f$, then $f$ and $f'$ are said to be equivalent.

**Lemma 2.5.** Let $f: M^n \to CP^m(c)$ be a Kaehler immersion of a connected complete Kaehler manifold $M^n$. Assume that $n \geq 2$ and $f$ is isotropic on a connected open subset $M_0$ in $M^n$. Then $M^n = CP^n(c/k)$ and $f$ is equivalent to $i \circ V^n_k$ for some $k$.

**Proof.** Using (1.2), we see that the holomorphic sectional curvature of $M_0$ is equal to $c - 2\lambda^2$ where $\lambda^2 = \|H(X,X)\|^2$. It follows from the holomorphic analogue of Schur’s Theorem [2, Theorem 7.5, p. 168] that $M_0$ ($n \geq 2$) is a Kaehler manifold of constant holomorphic sectional curvature. Thus $M^n$ is also of constant holomorphic sectional curvature since $M^n$ is analytic. Hence we can conclude from [3] that $M^n = CP^n(c/k)$ and $f$ is equivalent to $i \circ V^n_k$ for some positive integer $k$. Q.E.D.

**Theorem 2.6.** Let $f: M^n \to CP^m(c)$ be a proper 4-planar geodesic Kaehler immersion and $n \geq 2$. Then $M^n = CP^n(c/4)$ and $f$ is equivalent to $i \circ V^n_4$, where $i:CP^m(4)(c) \to CP^m(c)$ is a totally geodesic imbedding.

**Remark.** If $m(4) > m$, then such immersion does not exist.

**Proof.** Assume that the set $S$ in Lemma 2.2 is not empty. By Lemma 2.2, there are two cases: (I) $P_\pi X$ is totally real for every $X \in S$, and (II) $P_\pi X$ is complex for every $X \in S$.

Case (I). Equation (2.1) implies that

$$\langle JH(X,X), (DH)(X,X,X) \rangle = 0$$

for every $X \in S$. Since the left-hand side of (2.2) is real analytic on $UM$ and $S$ is open, we see that (2.2) holds for every $X \in UM$. Using (1.3) and the Codazzi equation, we have

$$\langle DH(JZ,Y,X), JH(X,X,X) \rangle = 0$$

for every $X, Y, Z \in TM$. Replacing $X$ by $JX$ in (2.2) and using (1.3) and (2.3), we have $\langle H(X,Y), (DH)(X,X,X) \rangle = 0$ for every $X \in TM$. Therefore, we conclude from Lemmas 2.3 and 2.4 that $M = CP^n(c/4)$ and $f$ is equivalent to $i \circ V^n_4$.

Case (II). Since $\phi(X) \neq 0$ on $S$, $H(X,X) \neq 0$ for every $X \in S$. Thus vectors $X, JX, H(X,X)$, and $JH(X,X)$ span $T_{\pi(X)}P_\pi X$. It follows from (2.1) that

$$\langle H(X,X), H(X,Y) \rangle = \langle A_H(X,X), X, Y \rangle = 0$$

for any $Y \in T_{\pi(X)}M$ orthogonal to Span$\{X, JX\}$ ($X \in S$). Furthermore, we have

$$\langle H(Y,X), H(X,JX) \rangle = \langle H(X,X), JH(X,X) \rangle = 0.$$

Therefore, (2.4) holds for every $Y \in T_{\pi(X)}M$ orthogonal to $X \in S$. In other words, the function $\lambda^2: X \mapsto \|H(X,X)\|^2$ defined on $UM$ has the vanishing derivative in the direction of the fibre on $U_{\pi(X)}M \cap S$ ($X \in S$). Since $U_{\pi(X)}M \cap S$ is open and $\lambda^2$...
is real analytic, $\lambda^2$ is constant on $U_{\pi(X)}M$ for $X \in S$. Thus $f$ is $\lambda$-isotropic on the connected open subset $M_0 = \pi(S)$. We see from Lemma 2.5 that $M^n = CP^n(c/k)$ and $f$ is equivalent to $i \circ V_k^n$ for some positive integer $k$. The $k$th Veronese map $V_k^n: CP^n(c/k) \to CP^m(k)(c)$ is proper $k$-planar geodesic. However, $P_\gamma$ is totally real for every geodesic $\gamma$ in $CP^n(c/k)$ (cf. [6, Lemma 2.2 and its proof, p. 303]). Hence this case does not occur.

Next let us assume that $\phi = 0$ on $UM$. Suppose that there exists a geodesic $\gamma$ such that $P_\gamma$ is totally real. Then the order of $f \circ \gamma$ is not greater than 3, and hence an open segment $(f \circ \gamma)(I)$ is contained in a 3-dimensional totally geodesic submanifold of $P_\gamma = RP^3(c/4)$ (for the definition of the order of a curve, see [6]). This contradicts the assumption that $f$ is proper 4-planar geodesic. Thus $P_\gamma$ is a complex totally geodesic submanifold for every $\gamma$. If $f$ is not totally geodesic, then the function $\lambda^2$ does not vanish identically on $UM$. Thus if we define $S$ by a connected component of the set $\{X \in UM, \lambda(X) \neq 0\}$ and $M_0 = \pi(S)$, then, using the same argument as Case (II), we have a contradiction. Q.E.D.

REMARK. We have used the condition that $f$ is 4-planar geodesic in order to prove $\langle \nabla^2_X \tau, Y \rangle = 0$ for every $Y$ orthogonal to $Sp\{X,JX\}$. There is a conjecture that if $f:M \to CP^m(c)$ is a proper $d$-planar geodesic Kaehler immersion, where $d$ is even, then $M = CP^m(c/d)$ and $f$ is equivalent to $V_d^n$.

It seems to be interesting that we characterize Kaehler immersions of compact symmetric Kaehler manifolds by the shape of geodesies.

The following is an easy consequence of Lemma 2.3.

**PROPOSITION 2.7.** Let $f:M \to CP^m(c)$ be a Kaehler immersion of a connected complete Kaehler manifold $M$. The first Frenet curvature of $\tau = f \circ \gamma$ is constant along $\tau$ for every geodesic $\gamma$ of $M$ if and only if $M$ is a compact simply connected symmetric Kaehler manifold and $f$ is equivalent to a full equivariant Kaehler imbedding mentioned before in this section.

**REFERENCES**


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