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4-PLANAR GEODESIC KAEHLER IMMERSIONS INTO A COMPLEX PROJECTIVE SPACE

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ABSTRACT. If f is a proper 4-planar geodesic Kaehler immersion of a connected complete Kaehler manifold M^n ($n \geq 2$) into $CP^m(c)$, then $M^n = CP^n(c/4)$ and f is equivalent to the 4th Veronese map.

0. Introduction. Let $\overline{M}$ be a Riemannian manifold. A curve $\tau: I \to \overline{M}$ defined on an open interval I is said to be *d-planar* if there exist an open interval I_s ($s \in I_s \subset I$) and a d-dimensional totally geodesic submanifold P_s for each $s \in I$ such that $\tau(I_s) \subset P_s$. Moreover, a d-planar curve τ is said to be *proper* if it is not $(d-1)$-planar on each open subinterval of I. An isometric immersion $f: M \to \overline{M}$ of a Riemannian manifold M is called a (resp. *proper*) *d-planar geodesic immersion* if $\tau = f \circ \gamma$ is (resp. proper) d-planar geodesic for every geodesic $\gamma: I \to M$. 1-planar geodesic immersions are totally geodesic. 2-planar geodesic immersions into real space forms were classified in [**7**] (for other treatment, see [**1**]).

When the ambient manifold is a complex projective space $CP^m(c)$ with constant holomorphic sectional curvature c, 2-planar and odd order proper planar geodesic Kaehler immersions were classified in [**5** and **6**], respectively. In this paper, we shall study proper 4-planar geodesic Kaehler immersions into $CP^m(c)$.

1. Notation and basic equations (cf. [2]). For a Kaehler immersion $f: M \to CP^m(c)$, the second fundamental form and Weingarten map corresponding to a normal vector field ξ will be denoted by H and A_ξ, respectively. Gauss and Weingarten's equations are given by

$$\overline{\nabla}_X Y = \nabla_X Y + H(X,Y), \qquad \overline{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi \tag{1.1}$$

for all tangent vector fields X and Y on M, where $\overline{\nabla}, \nabla$, and $\nabla^\perp$ denote the covariant differentiation of $\overline{M}, M$, and the normal bundle, respectively. Let R be the curvature tensor and J the complex structure of M. The structure equation of Gauss is given by

(1.2)
$$\begin{aligned} R(X,Y)Z &= (c/4)\{\langle Y,Z\rangle X - \langle X,Z\rangle Y + \langle JY,Z\rangle JX - \langle JX,Z\rangle JY - 2\langle JX,Y\rangle JZ\} \\ &\quad + A_{H(Y,Z)}X - A_{H(X,Z)}Y. \end{aligned}$$

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The structure equation of Codazzi reduces to $(DH)(X,Y,Z) = (DH)(Y,X,Z)$, where

$$(DH)(X,Y,Z) = \nabla^{\perp}_X H(Y,Z) - H(\nabla_X Y, Z) - H(Y, \nabla_X Z).$$

Let $\overline{J}$ be the complex structure of $CP^m(c)$. Since $\overline{\nabla}\overline{J} = 0$, we have

$$H(JX,Y) = \overline{J}H(X,Y). \tag{1.3}$$

If H satisfies $\|H(X,X)\|^2 = \lambda^2(x)$ for all unit vectors $X \in T_xM$ and each $x \in M$, then the immersion f is said to be *isotropic* (or λ-isotropic). We note that f is isotropic if and only if $\langle H(X,X), H(X,Y)\rangle = 0$ for any orthonormal vectors X and Y at every point.

2. Proper 4-planar geodesic Kaehler immersions. Let M be a connected complete Riemannian manifold and $f: M \to CP^m(c)$ a proper d-planar geodesic immersion. We first prove

LEMMA 2.1. *For each geodesic γ of M, there exists a unique d-dimensional totally geodesic submanifold P_γ such that $\tau((-\infty,\infty)) \subset P_\gamma$, where $\tau = f \circ \gamma$. Each P_γ is complex or totally real.*

PROOF. Let $u \in (-\infty,\infty)$ be arbitrarily fixed and put $P_\gamma = P_u$. If we define a set U by $U = \{s \in (-\infty,\infty): \tau(s) \in P_\gamma\}$, then U is nonempty and closed. Let $v \in U$. Consider a finite cover $\{I_{s_1} = I_u,\ I_{s_2}, \ldots, I_{s_k} = I_v\}$ of $[u,v]$ where we have assumed $u < v$ without loss of generality. Noting that f is proper d-planar geodesic and the intersection of two totally geodesic submanifolds is a totally geodesic submanifold, we see that $P_\gamma = P_u = P_{s_1} = \cdots = P_v$. Therefore, $I_v \subset U$, i.e., U is open and hence $U = (-\infty,\infty)$. The uniqueness of P_γ is easily derived from the assumption that f is proper d-planar geodesic. It is well known that a submanifold in $CP^m(c)$ is complex or totally real if and only if the second fundamental form H of the submanifold satisfies $(DH)(X,Y,Z) = (DH)(Y,X,Z)$ (cf. [**6**, (1.9), p. 300]). Therefore, since P_γ is totally geodesic, we have the assertion. Q.E.D.

Let $x \in M$, $X \in U_xM$ (unit tangent sphere at x), and let γ be the unit speed geodesic such that $\gamma(0) = x$ and $\dot\gamma(0) = X$. Then $\tau = f \circ \gamma$ satisfies

$$\begin{aligned}
&\dot\tau(0) = f_*X, \\
&\overline{\nabla}_X\dot\tau = H(X,X), \\
&\overline{\nabla}^2_X\dot\tau = -A_{H(X,X)}X + (DH)(X,X,X).
\end{aligned} \tag{2.1}$$

Higher order covariant derivatives of $\dot\tau$ in the direction X can be also obtained by using Gauss and Weingarten equations (1.1). Note that all covariant derivatives of $\dot\tau$ are tangent to P_γ. Define a function ϕ on the unit tangent sphere bundle UM over M by

$$\begin{aligned}
\phi(X) &= \det(\langle \overline{\nabla}^i_X\dot\tau, \overline{\nabla}^j_X\dot\tau\rangle_{i,j=0,1,\ldots,d-1}) \\
&= \text{Gramian of vectors } X, \overline{\nabla}_X\dot\tau, \ldots, \overline{\nabla}^{d-1}_X\dot\tau
\end{aligned}$$

for $X \in UM$. If $\phi(X) \neq 0$, then vectors $X, \overline{\nabla}_X\dot\tau, \ldots, \overline{\nabla}^{d-1}_X\dot\tau$ form a base of T_xP_γ.

LEMMA 2.2. *Let S be any connected component of the set $\{X \in UM: \phi(X) \neq 0\}$. Then P_{γ^X} is complex for every $X \in S$ or totally real for every $X \in S$ where γ^X denotes the geodesic tangent to X.*

PROOF. Assume that there exist X and Y in S such that P_{γ^X} is complex and P_{γ^Y} is totally real. Since S is arcwise connected, there is a smooth curve $X(t)$ in S such that $X(0) = X$ and $X(1) = Y$. Consider a function ψ on $[0,1]$ defined by

$$\psi(t) = \mathrm{Sup}\{\langle \overline{J}X(t), Z\rangle : Z \in T_{\pi(X(t))}P_{\gamma^{X(t)}}, \|Z\| = 1\},$$

where $\pi: UM \to M$ is the projection. Since

$$T_{\pi(X(t))}P_{\gamma^{X(t)}} = \mathrm{Span}\{X(t), \overline{\nabla}_{X(t)}\dot{\tau}_t, \ldots, \overline{\nabla}^{d-1}_{X(t)}\dot{\tau}_t\} \qquad (\tau_t = f \circ \gamma^{X(t)})$$

which is a smooth curve in the Grassmann bundle of d-planes over M, we see that ψ is a continuous function. Moreover, $P_{\gamma^{X(t)}}$ is complex or totally real (Lemma 2.1) and hence $\psi(t) = 1$ or 0 for each $t \in [0,1]$. Thus ψ is constant. However $\psi(0) = 1$ and $\psi(1) = 0$. Q.E.D.

Now we explain Kaehler immersions into $CP^m(c)$ of symmetric Kaehler manifolds of compact type. Let M be an irreducible symmetric Kaehler manifold of compact type and k a positive integer. In [**4**], Nakagawa and Takagi constructed a full equivariant Kaehler imbedding $f_k: M \to CP^m(c)$ which is called the kth full Kaehler imbedding of M. Moreover, in [**8**] Takagi and Takeuchi constructed a full Kaehler imbedding of a (not necessarily irreducible) symmetric Kaehler manifold M of compact type into $CP^m(c)$ as follows. Let M_i $(i = 1, \ldots, q)$ be the irreducible components of M, i.e., $M = M_1 \times \cdots \times M_q$ and $f_{k_i}: M_i \to CP^{m_i}(c)$ be the k_ith full Kaehler imbedding of M_i. Define a full Kaehler imbedding $S_q: CP^{m_1}(c) \times \cdots \times CP^{m_q}(c) \to CP^m(c)$ by the multifold tensor product of the homogeneous coordinates where $m = (m_1+1) \times \cdots \times (m_q+1) - 1$ and we notice that S_2 is the Segre imbedding. Then $S_q \circ (f_{k_1} \times \cdots \times f_{x_q})$ becomes a full equivariant Kaehler imbedding of M into $CP^m(c)$. In [**4** and **9**], it was shown that any full Kaehler immersion of a compact symmetric Kaehler manifold into $CP^m(c)$ is obtained in this way. In particular, if $M = CP^n(c/k)$, then the kth full Kaehler imbedding $V_k^n: CP^n(c/k) \to CP^{m(k)}(c)$ is called the *kth Veronese map* which is defined by

$$[z_i]_{0 \le i \le n} \mapsto \left[\left(\frac{k!}{k_0! \cdots k_n!}\right)^{1/2} z_0^{k_0} \cdots z_n^{k_n}\right]_{k_0 + \cdots + k_n = k},$$

where $[*]$ means the point of the projective space with the homogeneous coordinate $*$ and $m(k) = \binom{n+k}{k} - 1$.

The following two lemmas were proved in [**6**].

LEMMA 2.3 (CF. THE PROOF OF PROPOSITION 2.1 IN [**6**]). *Let $f: M \to CP^m(c)$ be a Kaehler immersion of a connected complete Kaehler manifold M into $CP^m(c)$. Assume that $\langle H(X,X), (DH)(X,X,X)\rangle = 0$ for every $X \in TM$. Then M is a compact simply connected symmetric Kaehler manifold.*

LEMMA 2.4 (CF. THE PROOF OF THEOREM 2.3 IN [**6**]). *Let $f: M \to CP^m(c)$ be a proper d-planar geodesic Kaehler immersion of a symmetric Kaehler* *manifold of compact type. Then* $M^n = CP^n(c/d)$ *and* f *is equivalent to* $i \circ V_d^n$, *where* $i: CP^{m(d)}(c) \to CP^m(c)$ *is a totally geodesic imbedding.*

Here we note that the equivalence of two isometric immersions f and f' of a Riemannian manifold into a Riemannian manifold $\overline{M}$ is defined as follows: If there exists an isometry F of $\overline{M}$ such that $f' = F \circ f$, then f and f' are said to be *equivalent.*

LEMMA 2.5. *Let* $f: M^n \to CP^m(c)$ *be a Kaehler immersion of a connected complete Kaehler manifold* M^n. *Assume that* $n \geq 2$ *and* f *is isotropic on a connected open subset* M_0 *in* M^n. *Then* $M^n = CP^n(c/k)$ *and* f *is equivalent to* $i \circ V_k^n$ *for some* k.

PROOF. Using (1.2), we see that the holomorphic sectional curvature of M_0 is equal to $c - 2\lambda^2$ where $\lambda^2 = \|H(X,X)\|^2$. It follows from the holomorphic analogue of Schur's Theorem [**2**, Theorem 7.5, p. 168] that M_0 ($n \geq 2$) is a Kaehler manifold of constant holomorphic sectional curvature. Thus M^n is also of constant holomorphic sectional curvature since M^n is analytic. Hence we can conclude from [**3**] that $M^n = CP^n(c/k)$ and f is equivalent to $i \circ V_k^n$ for some positive integer k. Q.E.D.

THEOREM 2.6. *Let* $f: M^n \to CP^m(c)$ *be a proper* 4*-planar geodesic Kaehler immersion and* $n \geq 2$. *Then* $M^n = CP^n(c/4)$ *and* f *is equivalent to* $i \circ V_4^n$, *where* $i: CP^{m(4)}(c) \to CP^m(c)$ *is a totally geodesic imbedding.*

REMARK. If $m(4) > m$, then such immersion does not exist.

PROOF. Assume that the set S in Lemma 2.2 is not empty. By Lemma 2.2, there are two cases: (I) $P_{\gamma X}$ is totally real for every $X \in S$, and (II) $P_{\gamma X}$ is complex for every $X \in S$.

Case (I). Equation (2.1) implies that

$$\langle \overline{J}H(X,X), (DH)(X,X,X) \rangle = 0 \tag{2.2}$$

for every $X \in S$. Since the left-hand side of (2.2) is real analytic on UM and S is open, we see that (2.2) holds for every $X \in UM$. Using (1.3) and the Codazzi equation, we have

$$(DH)(JZ,Y,X) = \overline{J}(DH)(Z,Y,X) \tag{2.3}$$

for every $X, Y, Z \in TM$. Replacing X by JX in (2.2) and using (1.3) and (2.3), we have $\langle H(X,X), (DH)(X,X,X) \rangle = 0$ for every $X \in TM$. Therefore, we conclude from Lemmas 2.3 and 2.4 that $M = CP^n(c/4)$ and f is equivalent to $i \circ V_4^n$.

Case (II). Since $\phi(X) \neq 0$ on S, $H(X,X) \neq 0$ for every $X \in S$. Thus vectors $X, JX, H(X,X)$, and $\overline{J}H(X,X)$ span $T_{\pi(X)}P_{\gamma X}$. It follows from (2.1) that

$$\langle H(X,X), H(X,Y) \rangle = \langle A_{H(X,X)}X, Y \rangle = 0 \tag{2.4}$$

for any $Y \in T_{\pi(X)}M$ orthogonal to $\mathrm{Span}\{X, JX\}$ ($X \in S$). Furthermore, we have

$$\langle H(X,X), H(X,JX) \rangle = \langle H(X,X), \overline{J}H(X,X) \rangle = 0.$$

Therefore, (2.4) holds for every $Y \in T_{\pi(X)}M$ orthogonal to $X \in S$. In other words, the function $\lambda^2: X \mapsto \|H(X,X)\|^2$ defined on UM has the vanishing derivative in the direction of the fibre on $U_{\pi(X)}M \cap S$ ($X \in S$). Since $U_{\pi(X)}M \cap S$ is open and λ^2

is real analytic, λ^2 is constant on $U_{\pi(X)}M$ for $X \in S$. Thus f is λ-isotropic on the connected open subset $M_0 = \pi(S)$. We see from Lemma 2.5 that $M^n = CP^n(c/k)$ and f is equivalent to $i \circ V_k^n$ for some positive integer k. The kth Veronese map $V_k^n: CP^n(c/k) \to CP^{m(k)}(c)$ is proper k-planar geodesic. However, P_γ is totally real for every geodesic γ in $CP^n(c/k)$ (cf. [**6**, Lemma 2.2 and its proof, p. 303]). Hence this case does not occur.

Next let us assume that $\phi = 0$ on UM. Suppose that there exists a geodesic γ such that P_γ is totally real. Then the order of $f \circ \gamma$ is not greater than 3, and hence an open segment $(f \circ \gamma)(I)$ is contained in a 3-dimensional totally geodesic submanifold of $P_\gamma = RP^3(c/4)$ (for the definition of the order of a curve, see [**6**]). This contradicts the assumption that f is proper 4-planar geodesic. Thus P_γ is a complex totally geodesic submanifold for every γ. If f is not totally geodesic, then the function λ^2 does not vanish identically on UM. Thus if we define S by a connected component of the set $\{X \in UM,\ \lambda(X) \neq 0\}$ and $M_0 = \pi(S)$, then, using the same argument as Case (II), we have a contradiction. Q.E.D.

REMARK. We have used the condition that f is 4-planar geodesic in order to prove $\langle \overline{\nabla}^2_X \dot{\tau}, Y \rangle = 0$ for every Y orthogonal to $\mathrm{Sp}\{X, JX\}$. There is a conjecture that if $f: M \to CP^m(c)$ is a *proper* d-planar geodesic Kaehler immersion, where d is even, then $M = CP^n(c/d)$ and f is equivalent to V_d^n.

It seems to be interesting that we characterize Kaehler immersions of compact symmetric Kaehler manifolds by the shape of geodesics.

The following is an easy consequence of Lemma 2.3.

PROPOSITION 2.7. *Let $f: M \to CP^m(c)$ be a Kaehler immersion of a connected complete Kaehler manifold M. The first Frenet curvature of $\tau = f \circ \gamma$ is constant along τ for every geodesic γ of M if and only if M is a compact simply connected symmetric Kaehler manifold and f is equivalent to a full equivariant Kaehler imbedding mentioned before in this section.*

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