PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 102, Number 4, April 1988

SOME STATIONARY SUBSETS OF $\mathcal{P}(\lambda)$

HANS-DIETER DONDER, PETER KOEPKE AND JEAN-PIERRE LEVINSKI

(Communicated by Thomas J. Jech)

ABSTRACT. Let κ and λ be *uncountable cardinals* such that $\kappa \leq \lambda$, and set $S(\kappa,\lambda) = \{X \in \mathcal{P}_\kappa(\lambda) | \, |X \cap \kappa| < |X|\}$. We determine the consistency strength of the statement "$(\exists\lambda \geq \kappa)(S(\kappa,\lambda)$ is stationary in $\mathcal{P}_\kappa(\lambda))$" using a new type of partition cardinals. In addition, we show that the property "$S(\kappa,\kappa^+)$ is stationary in $\mathcal{P}_\kappa(\kappa^+)$" is much stronger.

Let κ and λ be two cardinals such that κ is regular and $\lambda \geq \kappa > \omega$. Set $S(\kappa,\lambda) = \{X \in \mathcal{P}_\kappa(\lambda) | \, |X \cap \kappa| < |X|\}$. Baumgartner and Baldwin introduced this set (see [**2**]) and obtained partial results concerning the question whether $S(\kappa,\lambda)$ is stationary in $\mathcal{P}_\kappa(\lambda)$. In this paper we strengthen their results. We determine the exact consistency strength of the property "$(\exists\lambda \geq \kappa)(S(\kappa,\lambda)$ is stationary in $\mathcal{P}_\kappa(\lambda))$". Moreover, we show that this property becomes much stronger if we also require that the gap between κ and λ is small. Recall that if κ is κ^+-supercompact, then $S(\kappa,\kappa^+)$ is stationary in $\mathcal{P}_\kappa(\kappa^+)$ (see [**1**]).

1. Preliminaries. For our purposes it is useful to translate the statement "$S(\kappa,\lambda)$ is stationary" into a model-theoretic property which is just a variant of Chang's conjecture. Of course, this is already implicitly contained in [**2**]. Actually, we prove a more general result, which is part of the folklore.

Let λ be a cardinal such that $\lambda > \omega$. Let $\mathcal{M}_\lambda$ denote the set of all first-order structures $\mathfrak{A} = \langle \lambda, <, \dots \rangle$ of countable type. For $\mathfrak{A} \in \mathcal{M}_\lambda$, set $C_\mathfrak{A} = \{X \subset \lambda | X \prec \mathfrak{A}\}$. The proof of the following remark is left to the reader.

REMARK 1.1. Let $\mathcal{C}$ be any first-order structure of countable type such that $\lambda \subset \mathcal{C}$. Then, there is some $\mathfrak{A} \in \mathcal{M}_\lambda$ such that $C_\mathfrak{A} = \{X \cap \lambda | X \prec \mathcal{C}\}$.

Now, let κ be a *regular* cardinal such that $\omega < \kappa \leq \lambda$. Set $E(\kappa,\lambda) = \{X \in \mathcal{P}_\kappa(\lambda) | X \cap \kappa \in \kappa\}$.

LEMMA 1.2. *The club filter on $\mathcal{P}_\kappa(\lambda)$ is generated by the set*

$$\{C_\mathfrak{A} \cap E(\kappa,\lambda) | \mathfrak{A} \in \mathcal{M}_\lambda\}.$$

PROOF. Clearly, for every $\mathfrak{A} \in \mathcal{M}_\lambda$, $C_\mathfrak{A} \cap E(\kappa,\lambda)$ is a club subset of $\mathcal{P}_\kappa(\lambda)$. So, let D be an arbitrary club subset of $\mathcal{P}_\kappa(\lambda)$. We have to find some $\mathfrak{A} \in \mathcal{M}_\lambda$ such that $D \supset C_\mathfrak{A} \cap E(\kappa,\lambda)$. Set $\mathcal{C} = \langle H_{\lambda^+}, \in, D, \kappa \rangle$. By Remark 1.1 it suffices to show that $X \cap \lambda \in D$ for every $X \prec \mathcal{C}$ such that $X \cap \lambda \in E(\kappa,\lambda)$. We may assume w.l.o.g. that $|X| < \kappa$. Since $X \cap \kappa$ is transitive, we have $Y \subset X$ for every $Y \in D \cap X$. Hence, $X \cap \lambda = \bigcup(D \cap X)$. Moreover, $D \cap X$ is directed, since D is unbounded. But then $X \cap \lambda \in D$, since D is closed. □

Received by the editors December 4, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 03E55.
Key words and phrases. Partition cardinals, core model, stationary.

Hence, we get

COROLLARY 1.3. *Let κ and λ be cardinals such that κ is regular and $\omega < \kappa \le \lambda$. The following properties are then equivalent:*

(1) *$S(\kappa, \lambda)$ is stationary in $\mathcal{P}_\kappa(\lambda)$.*

(2) *For every $\mathfrak{A} \in \mathcal{M}_\lambda$ there is some $X \prec \mathfrak{A}$ such that $X \cap \kappa \in \kappa$ and $|X \cap \kappa| < |X| < \kappa$.*

For an infinite cardinal τ and an ordinal γ let $\tau^{(\gamma)}$ denote the γth cardinal successor of τ (hence $\tau^{(0)} = \tau$). Using this notation we set $S_\gamma(\kappa, \lambda) = \{X \in \mathcal{P}_\kappa(\lambda) | \operatorname{otp}(X) = |X \cap \kappa|^{(\gamma)}\}$.

LEMMA 1.4. *Let κ and λ be cardinals such that κ is regular and $\kappa \le \lambda$. Assume that λ is the least cardinal $\bar{\lambda} \ge \kappa$ such that $S(\kappa, \bar{\lambda})$ is stationary in $\mathcal{P}_\kappa(\bar{\lambda})$. Then $S_1(\kappa, \lambda)$ is stationary in $\mathcal{P}_\kappa(\lambda)$.*

PROOF. We shall show

$$(*) \qquad \begin{array}{l} \text{Let } X \prec \langle H_\lambda, < \rangle,\ \kappa \in X,\ X \cap \kappa \in \kappa,\ |X| < \kappa. \\ \text{Then } \operatorname{otp}(X) \le |X \cap \kappa|^+. \end{array}$$

Actually, this implies that, for some club $C \subset \mathcal{P}_\kappa(\lambda)$, $S(\kappa, \lambda) \cap C \subset S_1(\kappa, \lambda)$. So, we only have to show $(*)$. Let X be given and let $\bar{\lambda} \in X - \kappa$. By the minimality of λ, there is some $\mathfrak{A} \in \mathcal{M}_{\bar{\lambda}}$ which is a counterexample to property (2) of Corollary 1.3 (for κ, $\bar{\lambda}$). But, since $X \prec H_\lambda$, we may assume that $\mathfrak{A} \in X$. But then $X \cap \bar{\lambda} \prec \mathfrak{A}$. So $|X \cap \bar{\lambda}| = |X \cap \kappa|$, since $\mathfrak{A}$ is a counterexample. This shows that $\operatorname{otp}(X) \le |X \cap \kappa|^+$. □

There is an obvious generalization of Lemma 1.4 concerning the sets $S_\gamma(\kappa, \lambda)$ for $1 < \gamma < \kappa$. Moreover, giving a $\kappa > \omega$ we may define for $\gamma < \kappa$ $\lambda_\gamma(\kappa) \simeq$ the least $\lambda \ge \kappa$ such that $S_\gamma(\kappa, \lambda)$ is stationary in $\mathcal{P}_\kappa(\lambda)$. Using the argument above it is easy to see that if $\mu < \gamma < \kappa$ and $\lambda_\gamma(\kappa)$ exists then $\lambda_\mu(\kappa)$ exists and $\lambda_\mu(\kappa) < \lambda_\gamma(\kappa)$.

In [**1**] Baldwin showed that for a supercompact κ we have $\lambda_\gamma(\kappa) = \kappa^{(\gamma)}$ for all $\gamma < \kappa$.

2. More Erdös cardinals. In order to formulate the equiconsistency result promised at the beginning we introduce a new type of Erdös cardinals.

DEFINITION. Let κ and λ be two cardinals such that κ is *weakly inaccessible* and $\lambda > \kappa$. Then λ is κ-*Baldwin* iff for every club set C in λ and every regressive function $f\colon [C]^{<\omega} \to \lambda$, there exists an infinite set $I \subset C$ such that I is f-homogeneous and such that $|I| > \sup(\kappa \cap f''[I]^{<\omega})$.

Of course, the definitioin could be given for arbitrary κ. But, for example, if $\kappa = \tau^+$, this would be equivalent to being κ-Erdös. On the other hand, if $\operatorname{cof}(\kappa) = \omega$, it would be equivalent to being κ^+-Erdös. Therefore, we restrict this definition to weakly inaccessible κ. Clearly, in that case every κ-Erdös cardinal λ such that $\lambda > \kappa$ is κ-Baldwin. But we shall note that the converse is not true. As usual, the combinatorial definition is equivalent to a model-theoretic one.

DEFINITION. Let $\mathfrak{A} = \langle L_\lambda[\overrightarrow{A}], \in, A_1, \ldots, A_n \rangle$ be a structure and $I \subset \lambda$. Then I is a *good set of indiscernibles* for $\mathfrak{A}$ iff, for all $\gamma \in I$, $L_\gamma[\overrightarrow{A}] \prec \mathfrak{A}$ and $I - \gamma$ is a set of indiscernibles for $\langle \mathfrak{A}, (\xi)_{\xi < \gamma} \rangle$.

For a structure $\mathfrak{A}$ and $I \subset \mathfrak{A}$ let $\mathrm{Hull}_{\mathfrak{A}}(I)$ be the set of all elements of $\mathfrak{A}$ which are definable in $\mathfrak{A}$ with parameters from I. The proof of the following lemma is standard (cf. Lemma 17.12 in [**4**]), and therefore left to the reader.

LEMMA 2.1. *Let κ and λ be two cardinals such that $\lambda > \kappa$. Then λ is κ-Baldwin iff every $\mathfrak{A} = \langle L_\lambda[\overrightarrow{A}], \in, \overrightarrow{A}\rangle$ has a good set of indiscernibles I such that $|I| > \sup(\kappa \cap \mathrm{Hull}_{\mathfrak{A}}(I))$.*

If κ and λ are cardinals s.t. κ is weakly inaccessible and $\lambda < \kappa$, we have

$$\lambda \text{ is } \kappa\text{-Erdös} \rightarrow \lambda \text{ is } \kappa\text{-Baldwin} \rightarrow \lambda \text{ is } \alpha\text{-Erdös}$$

for all $\alpha < \kappa$. We now show that none of the arrows can be reversed.

LEMMA 2.2. *Let κ and λ be cardinals such that $\lambda > \kappa$ and κ is weakly inaccessible.*

(a) *If λ is κ-Baldwin, then there is some τ such that $\kappa < \tau < \lambda$ and τ is α-Erdös for all $\alpha < \kappa$.*

(b) *If λ is κ-Erdös, then there is some τ such that $\kappa < \tau < \lambda$ and τ is κ-Baldwin.*

PROOF. (a) Let λ be κ-Baldwin and assume that the conclusion is false. Since λ is inaccessible, there is some $A \subset \lambda$ such that $H_\lambda = L_\lambda[A]$. Set $\mathfrak{A} = \langle L_\lambda[A], \in, A, \{\kappa\}\rangle$. Let $I \neq \emptyset$ be a good set of indiscernibles for $\mathfrak{A}$. Then $\kappa < \min(I)$. So, by our assumption, for $\gamma \in I$ let d_γ be the least $\alpha < \kappa$ such that γ is not α-Erdös and let $\mathfrak{A}_\gamma$ be the $\mathfrak{A}$-least witness of this fact. Then d_γ is the same for all $\gamma \in I$. Set $\alpha = d_\gamma$. Moreover, $I \cap \gamma$ is a good set of indiscernibles for $\mathfrak{A}_\gamma$ if $\gamma \in I$. Hence, $\mathrm{ot}(I) \leq \alpha$. But $\alpha \in \mathrm{Hull}_{\mathfrak{A}}(I) \cap \kappa$. So $\mathfrak{A}$ shows that λ is not κ-Baldwin, which is a contradiction.

(b) Define $\mathfrak{A}$ as in (a) and let $I \subset \lambda$, $\mathrm{otp}(I) = \kappa$, be a good set of indiscernibles for $\mathfrak{A}$. Set $\beta = \sup(\kappa \cap \mathrm{Hull}_{\mathfrak{A}}(I))$. Then $\beta < \kappa$. Since κ is a limit cardinal there is some $\gamma \in I$ such that $|I \cap \gamma| > \beta$. But then it is easy to see that γ is κ-Baldwin. □

Actually, in (b) it is sufficient to assume that λ is almost $< \kappa$-Erdös (see [**6**] for a definition of this notion).

The next result somehow improves Theorem 4 of [**2**].

THEOREM 2.3. *Let λ be κ-Baldwin, where κ is weakly inaccessible and $\lambda > \kappa$. Then $S_1(\kappa, \lambda)$ is stationary in $\mathcal{P}_\kappa(\lambda)$.*

PROOF. By the results of §1 it suffices to show that for every $\mathfrak{A} \in \mathcal{M}_\lambda$ there is some $X \prec \mathfrak{A}$ such that $X \cap \kappa \in \kappa$ and $\mathrm{otp}(X) = |X \cap \kappa|^+$. So let $\mathfrak{A} \in \mathcal{M}_\lambda$ be given. Choose some $A \subset \lambda$ such that $\mathfrak{A}$ is coded in $\mathcal{C} = \langle L_\lambda[A], \in, A, \kappa\rangle$. Let $I \subset \lambda$ be a good set of indiscernibles for $\mathcal{C}$ such that $|I| > \sup(\kappa \cap \mathrm{Hull}_{\mathcal{C}}(I))$. Set $\tau = \sup(\kappa \cap \mathrm{Hull}_{\mathcal{C}}(I))$. Hence $\tau < \kappa$. Let $\bar{I}$ be the initial segment of I such that $\mathrm{otp}(\bar{I}) = \tau^+$. Now set $Y = \mathrm{Hull}_{\mathcal{C}}(\tau \cup \bar{I})$. Then $Y \prec \mathcal{C}$, and standard indiscernibility arguments show that $\tau = \kappa \cap Y$ and $\mathrm{otp}(Y \cap \lambda) = \tau^+$. So, because $\mathfrak{A}$ is coded in $\mathcal{C}$, we have, setting $X = Y \cap \lambda$, that $X \prec \mathfrak{A}$, $X \cap \kappa \in \kappa$ and $\mathrm{otp}(X) = |X \cap \kappa|^+$. □

For the next result we have to assume acquaintance with the basic properties of the Dodd-Jensen core model K (see [**4**]).

THEOREM 2.4. *If $S_1(\kappa, \lambda)$ is stationary in $\mathcal{P}_\kappa(\lambda)$, then λ is κ-Baldwin in K.*

PROOF. We shall show that λ satisfies the condition from Lemma 2.1 in K. First, note that κ is *weakly inaccessible* (see Theorem 3 in [**2**]). Now let $\mathfrak{A} = \langle L_\lambda[\overrightarrow{A}], \in, \overrightarrow{A}\rangle \in K$ be a structure. We may assume w.l.o.g. that $L_\lambda[\overrightarrow{A}] = K_\lambda$.

Since $S_1(\kappa, \lambda)$ is stationary, there is some $X \prec H_{\lambda^+}$, with $\mathfrak{A} \in X$ and some $\tau < \kappa$ such that $X \cap \kappa = \tau$ and $\text{otp}(X \cap \lambda) = \tau^+$. Let $\bar{H}$ be transitive and $\sigma: \bar{H} \xrightarrow{\sim} X$ be an isomorphism. Set $\sigma(\bar{K}) = K_\lambda$, $\sigma(\bar{\mathfrak{A}}) = \mathfrak{A}$. If $\bar{K} = K_{\tau^+}$, then σ shows that some ordinal $\rho \leq \tau^+$ is measurable in an inner model (see Claim 1.5 in [**6**]). But then λ is measurable in an inner model, hence λ is Ramsey in K. So we may assume that $\bar{K} \neq K_{\tau^+}$. But then there is some mouse M such that $|M| \leq \tau$ and $M \notin \bar{K}$. Then M is bigger than every mouse $\bar{M} \in \bar{H}$. Since $\bar{H} \models \bar{\mathfrak{A}} \in K$ it follows that $\bar{\mathfrak{A}} \in N$, where N is the τ^+th mouse iterate of M. But then some final segment $C \subset \tau^+ - \tau$ of the first τ^+ iteration points of M is a good set of indiscernibles for $\bar{\mathfrak{A}}$. Now set $\bar{I} = \sigma''C$. Then $\bar{I}$ is a good set of indiscernibles for $\mathfrak{A}$. Note that $\text{otp}(\bar{I}) = \text{otp}(C) = \tau^+$. Hence, by Jensen's indiscernibles Lemma (see [**4** or **5**]) there is some $I \in K$ such that $\bar{I} \subset I$ and I is a good set of indiscernibles for $\mathfrak{A}$. We may assume that $\kappa \leq \min(I)$. But then $\text{Hull}_{\mathfrak{A}}(I) \cap \kappa = \text{Hull}_{\mathfrak{A}}(\bar{I}) \cap \kappa \subset X \cap \kappa = \tau$. So I is as required. □

By Lemma 1.4 the last two theorems give the promised equiconsistency results concerning the property "$\exists \lambda \geq \kappa$, $S(\kappa, \lambda)$ is stationary in $\mathcal{P}_\kappa(\lambda)$". There is an obvious generalization of these results concerning the sets $S_\gamma(\kappa, \lambda)$ for $1 < \gamma < \kappa$. We leave this to the interested reader (note that when $\text{cof}(\gamma) = \omega$, an additional argument is needed).

3. Small gaps. Let us call a set $X \subset \mathcal{P}(\lambda)$ *stationary in* $\mathcal{P}(\lambda)$ if, for all $\mathfrak{A} \in \mathcal{M}_\lambda$, $X \cap C_{\mathfrak{A}} \neq \emptyset$, where $\mathcal{M}_\lambda$ and $C_{\mathfrak{A}}$ are defined as in §1. The next result uses only arguments contained in [**6** and **7**]. So we only sketch the proof.

THEOREM 3.1. *Let $\lambda > \omega_1$ be a cardinal which is not weakly Mahlo. Let $S = \{X \subset \lambda | X \notin \text{On}, |X| > \omega_1, \text{otp}(X)$ is a cardinal, $\text{cf}(\text{otp}(X)) > \omega\}$ be stationary in $\mathcal{P}(\lambda)$. Then there is an inner model with a measurable cardinal.*

PROOF. Assume the conclusion is false. Then the core model K covers V. We shall use this in order to derive a contradiction. Choose some $X \prec H_{\lambda^+}$ such that $X \cap \lambda \in S$. Let $\sigma: \bar{H} \xrightarrow{\sim} X$ be an isomorphism, with $\bar{H}$ transitive, and let $\sigma(\tau) = \lambda$, $\sigma(\bar{K}) = K_\lambda$. Since $X \cap \lambda \in S$ we know that τ is a cardinal, $\tau > \omega_1$, $\text{cf}(\tau) > \omega$, and $\sigma|\tau \neq \text{id}|\tau$. So it suffices to show that $\bar{K} = K_\tau$, since then σ induces a nontrivial elementary embedding of K into K which contradicts our assumption. Now, by a version of Lemma 2.7 in [**6**] it suffices to show that for every club $C \subset \tau$ there is some $\gamma \in C$ which is singular in $\bar{K}$. If τ is a limit cardinal in $\bar{H}$, then, by the covering lemma, there is some club $B \subset \tau$, $B \in \bar{H}$, such that every $\gamma \in B$ is singular in $\bar{K}$. Here we used that τ is not Mahlo in $\bar{H}$. So we get $\bar{K} = K_\tau$ because $\text{cof}(\tau) > \omega$. If τ is a successor cardinal in $\bar{H}$, then, since $\bar{H} \models \tau > \omega_2$, there is some $\delta < \tau$ such that all ordinals γ, $\delta < \gamma < \tau$, which are regular in $\bar{K}$ have the same cofinality in $\bar{H}$. But in this case τ is a successor cardinal in V and $\tau > \omega_1$. So we also get $\bar{K} = K_\tau$. □

Now, by Lemma 1.2, we get

COROLLARY 3.2. *Let $\kappa > \omega$ be regular and let $\lambda \geq \kappa$ be a cardinal which is not weakly Mahlo. Assume that $S_1(\kappa, \lambda)$ is stationary in $\mathcal{P}_\kappa(\lambda)$. Then there is an inner model with a measurable cardinal.*

By Lemma 1.4, this gives a lower bound on the consistency of "$S(\kappa, \kappa^+)$ is stationary in $\mathcal{P}_\kappa(\kappa^+)$" and a negative answer to the last question of [**2**] (see also [**7**, Theorem 11]). Using higher core models, this bound can be improved.

Finally, let us mention that Theorem 3.1 is also relevant for the questions treated in [**3**].

REFERENCES

1. S. Baldwin, *Generalizing the Mahlo hierarchy, with applications to the Mitchell models*, Ann. Pure Appl. Logic **25** (1983), 103–127.
2. ——, *The consistency strength of certain stationary subsets of $\mathcal{P}_\kappa(\lambda)$*, Proc. Amer. Math. Soc. **92** (1984), 90–92.
3. C. A. Di Prisco and W. Marek, *A filter on $[\lambda]^\kappa$*, Proc. Amer. Math. Soc. **90** (1984), 591–598.
4. A. J. Dodd, *The core model*, London Math. Soc. Lecture Notes, no. 61, Cambridge Univ. Press, 1982.
5. H.-D. Donder, R. B. Jensen and B. Koppelberg, *Some applications of K*, Set Theory and Model Theory (Proc. Bonn 1979), Lecture Notes in Math., vol. 872, Springer-Verlag, Berlin and New York, 1981, pp. 55–97.
6. H.-D. Donder and P. Koepke, *On the consistency strength of "accessible" Jonsson cardinals and of the weak Chang conjecture*, Ann. Pure Appl. Logic **25** (1983), 233–261.
7. J.-P. Levinski, *Instances of the conjecture of Chang*, Israel J. Math. **48** (1984), 225–243.

DEPARTMENT OF MATHEMATICS, FREIE UNIVERSITÄT BERLIN, 1000 BERLIN 33, FEDERAL REPUBLIC OF GERMANY (Current address of H.-D. Donder)

DEPARTMENT OF MATHEMATICS, WOLFSON COLLEGE, OXFORD, ENGLAND

Current address (Peter Koepke): Mathematisches Institut, Universität Freiburg, 7800 Freiburg, Federal Republic of Germany

Current address (J.-P. Levinski): Department of Mathematics, Dartmouth College, Hanover, New Hampshire 03755