ON QUASI-MINIMAL $e$-DEGREES
AND TOTAL $e$-DEGREES

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ABSTRACT. We show that there exists a set $A$ such that the $e$-degree of $A$ is
quasi-minimal and the $e$-degree of the complement of $A$ is total. This provides
also a counterexample to a conjecture in [1].

1. In [1, p. 426], J. Case conjectures that there are no sets $A$ such that $A$ lies
in a total $e$-degree and $ar{A}$ lies in a nontotal $e$-degree. We show in this note that
this conjecture is false: in fact there exist sets $A$ such that $[A]_e$ is quasi-minimal and
$[ar{A}]_e$ is total.

Our notation is standard as in [4], with a few changes and additions. $\omega$ de-
notes the set of natural numbers; if $A \subseteq \omega$, then $\bar{A}$ denotes the set $\omega \setminus A$.
$\{ (x, y) | x, y \in \omega \}$ is a one-one recursive coding of the pairs of natural numbers
onto $\omega$ (we assume $(0, 0) = 0$); for every $x, y \in \omega$, $(\langle x, y \rangle)_0 = x$. $\Phi_z$ is the en-
numeration operator defined through the recursively enumerable set $W_z$, i.e. $\Phi_z(A) =
\{ x | (\exists u)((x, u) \in W_z \cup D_u \subseteq A) \}$, where $D_u$ is the finite set having canonical
index $u$; as it is known, for every $A, B \subseteq \omega$, $A \leq_e B$ if and only if $(\exists z)(\Phi_z(B) = A)$.

If $\{ W^s_z \}_{s \in \omega}$ is a finite recursive approximation to $W_z$ (i.e. $W_z = \bigcup_{s \in \omega} W^s_z$)
and $(\forall s)[W^s_z \text{ finite} \& W^s_z \subseteq W^{s+1}_z]$ then $\Phi^*_z$ denotes the enumeration operator
given by $\Phi^*_z(A) = \{ x | (\exists u)((x, u) \in W^s_z \cup D_u \subseteq A) \}$. Let $A \subseteq \omega$: $A$ is single-valued
if $(\forall x)(\forall y)(\forall z) [(x, y) \in A \& (x, z) \in A \Rightarrow y = z]$; $A$ is total if $(\forall (x, y))[(x, y) \in A]$;
$[A]_e$ denotes the $e$-degree (also called partial degree) of $A$. An $e$-degree $a_e$ is total if, for
some function $f$, $\tau(f) \in a_e$ (where $\tau(f) = \{ (x, y) | (x, y) \in f \}$); $a_e$ is quasi-minimal if $a_e$
does not contain any recursively enumerable set and, for every function $f$ and for every set $A \in a_e$,
if $\tau(f) \leq_e A$ then $f$ is recursive.

The reader is referred to [4] for the definitions of the classes $\Sigma_n, \Pi_n, \Delta_n$ of the
arithmetic hierarchy as well as for the definition of one-one reducibility (notation:
$\leq_1$).

2. Let $K$ be a creative set; for example, let $K = \{ x | x \in W_z \}$.

LEMMMA. (a) $[\bar{K}]_e$ is total; (b) for every $A \in \Sigma_2$, $A \leq_e \bar{K}$.
PROOF. (a) That $[\mathcal{K}]_e$ is total follows from the fact that $\mathcal{K} \in \Pi_1$ and, for every $A \in \Pi_1$, $[A]_e$ is total (see Lemma 6(2) of [2]).

(b) By Corollary 3.1 and Proposition 2.1 of [3], for every $A \in \Sigma_2$,

$$A \leq e \{x|x \in \Phi_x(\emptyset)\}.$$ 

The claim follows since clearly $\{x|x \in \Phi_x(\emptyset)\} \leq_1 \mathcal{K}$; thus $\{x|x \in \Phi_x(\emptyset)\} \leq_1 \mathcal{K}$ and a fortiori $\{x|x \in \Phi_x(\emptyset)\} \leq_1 \mathcal{K}$. Q.E.D.

THEOREM. There exists a set $A$ such that $[A]_e$ is quasi-minimal and $[A]_e$ is total.

PROOF. The proof aims to construct a $\Delta_2$-set $A$ such that $K = \{y|0, y \in A\}$ and satisfying, for every $e \in \omega$, the requirements $\Phi_e(A)$ total $\Rightarrow$ $\Phi_e(A)$ recursively enumerable, and $A \neq W_e$. In order to satisfy the requirements $P_e$: $\Phi_e(A)$ total $\Rightarrow$ $\Phi_e(A)$ recursively enumerable, we put enough elements into $A$ to make $\Phi_e(A)$ non-single-valued. To ensure that $A$ is not recursively enumerable, we define by approximations a one-one sequence $\{m_e\}_{e \in \omega}$ such that, for every $e$, $m_e \in A \Leftrightarrow m_e \notin W_e$. Then we put into $A$ enough elements to make $\Phi_e(A)$ non-single-valued.

Let $K^s = \{K^s_s\}_{s \in \omega}$ be a finite recursive approximation to $K$ and, for every $z \in \omega$, let $\{\nu_s\}_{s \in \omega}$ be the standard enumeration of $W_z$.

We define a sequence $\{A^s\}_{s \in \omega}$ of finite sets by induction as follows.

Step (0). Let $A^0 = \emptyset$.

Step (s + 1). By induction on $e$, define the following set and functions:

$$H_e = \{x|(\exists i < e)[x = r(i, s)]\},$$

$$u(e, s) = \begin{cases} 
\mu u \leq s \cdot \{x \in D_u(x_0 > 0) \cap H_e \supseteq \emptyset \} & \text{such a } u \text{ exists,} \\
0, & \text{otherwise,}
\end{cases}$$

$$m(e, s) = \mu x \cdot (x_0 > 0 \land \forall t \leq s)[x > \max u(i, t)],$$

$$r(e, s) = \begin{cases} 
\mu x \cdot [x \in W^s_e \land (x_0 > 0 \land x \geq m(e, s))] & \text{if such a number exists,} \\
0, & \text{otherwise.}
\end{cases}$$

Then define $A^{s+1}$ by letting $x \in A^{s+1}$ if for some $e \leq s$ one of the following holds:

(a) $(\exists y)[x = (0, y) \land y \in K^s]$;

(b) $(\exists t \leq s)[x \in D_{u(e, t)}(\forall i < e)[(x_0 > 0 \Rightarrow x \neq r(i, s)]]$;

(c) $r(e, s) = 0 \land x = m(e, s)$.

This ends Step (s + 1).

Now define $A$ by $x \in A$ if $(\exists t)(\forall s \geq t)[x \in A^s]$; since the relation $x \in A^s$ is recursive, $A$ is clearly a $\Sigma_2$-set.

SUBLEMMA 1. $(\forall e)[\lim s u(e, s), \lim s m(e, s), \lim s r(e, s) \text{ exist}]$.
Proof of Sublemma 1. By induction. Let \( e \in \omega \) be given and assume that \( (\forall i < e) [\lim_s u(i,s), \lim_s m(i,s), \lim_s r(i,s) \text{ exist}] \): for every \( i < e \), let \( r_i = \lim_s r(i,s), m_i = \lim_s m(i,s) \) and let \( H_e = \{ x | (\exists i < e) [x = r_i] \} \). It easily follows that
\[
\lim_s u(e,s) = \begin{cases} 
\mu u \cdot \{ x \in D_u | (x)_0 > 0 \} \cap H_e = \emptyset \& \{ x | (0,x) \in D_u \} \subseteq K \& \\
\Phi_e(D_u) \text{ non-single-valued}, \quad \text{if such a } u \text{ exists,} \\
0, \quad \text{otherwise.}
\end{cases}
\]
Thus, letting \( D = \{ x | (\exists i \leq e)(\exists t) [x \in D_u(t,i)] \} \), since \( D \) is finite we have that \( \lim_s m(e,s) = \mu x \cdot \{ (x)_0 > 0 \& x > \max D \& (\forall i < e) [x > m_i \& x > r_i] \} \). The proof of Sublemma 1 is now complete since clearly \( \lim_s m(e,s) \) exists \( \Rightarrow \lim_s r(e,s) \) exists. Q.E.D.

For every \( e \in \omega \), let \( m_e = \lim_s m(e,s) \) and \( r_e = \lim_s r(e,s) \). Notice that if \( r_e \neq 0 \) then for every \( i > e \), \( r_e \in H_i \) and for every \( i < e \), \( r_e \neq m_i \) hence for cofinitely many \( s \), \( r_e \notin A^s \) and thus \( r_e \notin A \).

Sublemma 2. \( (\forall e) [\Phi_e(A) \text{ total } \Rightarrow \Phi_e(A) \text{ recursively enumerable}] \).

Proof of Sublemma 2. Suppose that \( \Phi_e(A) \) is total. Let
\[
K_0 = \{ x | (\exists y \in K) [x = (0,y)] \}
\]
and let
\[
H'_e = H_e \cap \{ x | (x)_0 > 0 \}.
\]
We claim that \( \Phi_e(A) = \Phi_e(K_0 \cup H'_e) \).

Indeed, that \( \Phi_e(A) \) is included in \( \Phi_e(K_0 \cup H'_e) \) is a consequence of the fact that \( A \subseteq K_0 \cup H'_e \). Suppose now that for some \( (y,v) \in \omega \), \( (y,v) \notin \Phi_e(A) \); since \( \Phi_e(A) \) is total, there exists \( w \neq v \) such that \( (y,w) \in \Phi_e(A) \). But, then, for some finite set \( D \subseteq A \), we have \( (y,w) \in \Phi_e(D) \) and therefore there exists a finite set \( E \) such that \( E \subseteq K_0 \cup H'_e \) and \( (y,v), (y,w) \in \Phi_e(E) \), i.e. \( \Phi_e(E) \) is not single-valued. The construction ensures that in this case \( \Phi_e(A) \) is not single-valued, contradicting the assumption that \( \Phi_e(A) \) is total. We have shown that if \( \Phi_e(A) \) is total then \( \Phi_e(A) = \Phi_e(K_0 \cup H'_e) \) but the latter set is manifestly recursively enumerable and thus the sublemma is proved. Q.E.D.

Sublemma 3. \( (\forall e) [A \neq W_e] \).

Proof of Sublemma 3. Let \( e \in \omega \) be given. We distinguish two cases.

Case 1. \( (\exists x) [x \in W_e \& (x)_0 > 0 \& x \geq m_e] \). In this case, \( r_e \) equals the least such \( x \): thus \( r_e \in W_e \) but, as already remarked, \( r_e \notin A \) and therefore \( A \neq W_e \) as desired.

Case 2. Otherwise. In this case, \( (\forall x) [x \in W_e \& (x)_0 > 0 \Rightarrow x < m_e] \) and, for cofinitely many \( s \), \( m_e \in A^s \); thus \( m_e \in A - W_e \) and the proof is complete. Q.E.D.

We are now in a position to conclude the proof of the theorem. Indeed, Sublemma 2 and Sublemma 3 ensure that \( [A]_e \) is quasi-minimal.

By Sublemma 1 we have that \( A \) is a \( \Delta_2 \)-set; moreover, \( (\forall y) [y \in K \Rightarrow (0,y) \in A] \). Thus \( K \leq_1 A \) (which implies \( \bar{K} \leq_e \bar{A} \)) and, by Lemma (b), also \( \bar{A} \leq_e \bar{K} \), since \( \bar{A} \in \Delta_2 \). Therefore \( \bar{A} \equiv_e \bar{K} \) and, by Lemma (a), \( [\bar{A}]_e \) is total.
REFERENCES


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