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ON QUASI-MINIMAL e-DEGREES AND TOTAL e-DEGREES

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ABSTRACT. We show that there exists a set A such that the e-degree of A is quasi-minimal and the e-degree of the complement of A is total. This provides also a counterexample to a conjecture in [**1**].

1. In [**1**, p. 426], J. Case conjectures that there are no sets A such that A lies in a total e-degree and $\bar{A}$ lies in a nontotal e-degree. We show in this note that this conjecture is false: in fact there exist sets A such that $[A]_e$ is quasi-minimal and $[\bar{A}]_e$ is total.

Our notation is standard as in [**4**], with a few changes and additions. ω denotes the set of natural numbers; if $A \subseteq \omega$, then $\bar{A}$ denotes the set $\omega - A$. $\{\langle x,y\rangle \mid x,y \in \omega\}$ is a one-one recursive coding of the pairs of natural numbers onto ω (we assume $\langle 0,0\rangle = 0$); for every $x,y \in \omega$, $(\langle x,y\rangle)_0 = x$. Φ_z is the enumeration operator defined through the recursively enumerable set W_z, i.e. $\Phi_z(A) = \{x \mid (\exists u)[\langle x,u\rangle \in W_z \,\&\, D_u \subseteq A]\}$, where D_u is the finite set having canonical index u; as it is known, for every $A,B \subseteq \omega$, $A \leq_e B$ if and only if $(\exists z)[\Phi_z(B) = A]$. If $\{W_z^s\}_{s\in\omega}$ is a finite recursive approximation to W_z (i.e. $W_z = \bigcup_{s\in\omega} W_z^s$ and $(\forall s)[W_z^s$ finite & $W_z^s \subseteq W_z^{s+1}])$ then Φ_z^s denotes the enumeration operator given by $\Phi_z^s(A) = \{x \mid (\exists u)[\langle x,u\rangle \in W_z^s \,\&\, D_u \subseteq A]\}$. Let $A \subseteq \omega$: A is *single-valued* if $(\forall x)(\forall y)(\forall z)[\langle x,y\rangle \in A \,\&\, \langle x,z\rangle \in A \Rightarrow y = z]$; A is *total* if $\{(x,y) \mid \langle x,y\rangle \in A\}$ is a total function from ω into ω (briefly a *function*, dropping the word "total" and the phrase "from ω into ω"); $[A]_e$ denotes the e-degree (also called partial degree) of A. An e-degree $\underline{a}_e$ is *total* if, for some function f, $\tau(f) \in \underline{a}_e$ (where $\tau(f) = \{\langle x,y\rangle \mid (x,y) \in f\}$); $\underline{a}_e$ is *quasi-minimal* if $\underline{a}_e$ does not contain any recursively enumerable set and, for every function f and for every set $A \in \underline{a}_e$, if $\tau(f) \leq_e A$ then f is recursive.

The reader is referred to [**4**] for the definitions of the classes $\Sigma_n, \Pi_n, \Delta_n$ of the arithmetical hierarchy as well as for the definition of one-one reducibility (notation: $\leq_1$).

2. Let K be a creative set; for example, let $K = \{x \mid x \in W_x\}$.

LEMMA. (a) $[\bar{K}]_e$ *is total*; (b) *for every* $A \in \Sigma_2$, $A \leq_e \bar{K}$.

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PROOF. (a) That $[\bar{K}]_e$ is total follows from the fact that $\bar{K} \in \Pi_1$ and, for every $A \in \Pi_1$, $[A]_e$ is total (see Lemma 6(2) of [**2**]).

(b) By Corollary 3.1 and Proposition 2.1 of [**3**], for every $A \in \Sigma_2$,

$$A \leq_e \overline{\{x|x \in \Phi_x(\emptyset)\}}.$$

The claim follows since clearly $\{x|x \in \Phi_x(\emptyset)\} \leq_1 K$; thus $\overline{\{x|x \in \Phi_x(\emptyset)\}} \leq_1 \bar{K}$ and a fortiori $\overline{\{x|x \in \Phi_x(\emptyset)\}} \leq_e \bar{K}$. Q.E.D.

THEOREM. *There exists a set A such that $[A]_e$ is quasi-minimal and $[\bar{A}]_e$ is total.*

PROOF. The proof aims to construct a Δ_2-set A such that $K = \{y|\langle 0,y\rangle \in A\}$ and satisfying, for every $e \in \omega$, the requirements $\Phi_e(A)$ total $\Rightarrow \Phi_e(A)$ recursively enumerable, and $A \neq W_e$. In order to satisfy the requirements P_e: $\Phi_e(A)$ total $\Rightarrow \Phi_e(A)$ recursively enumerable, we put enough elements into A to make $\Phi_e(A)$ non-single-valued. To ensure that A is not recursively enumerable, we define by approximations a one-one sequence $\{m_e\}_{e\in\omega}$ such that, for every e, $m_e \in A \Leftrightarrow m_e \notin W_e$: if $m_e \in W_e$ then the construction makes sure that $m_e \notin A$ by choosing m_e different from all the elements which are put into A in order to satisfy those P_i's for which $i \leq e$ and not allowing m_e to be put into A because of any P_i, $i > e$.

Let $\{K^s\}_{s\in\omega}$ be a finite recursive approximation to K and, for every $z \in \omega$, let $\{W_z^s\}_{s\in\omega}$ be the standard enumeration of W_z.

We define a sequence $\{A^s\}_{s\in\omega}$ of finite sets by induction as follows.

Step (0). Let $A^0 = \emptyset$.

Step (s+1). By induction on e, define the following set and functions:

$$H_e^s = \{x|(\exists i < e)[x = r(i,s)]\},$$

$$u(e,s) = \begin{cases} \mu u \leq s \cdot [\{x \in D_u|(x)_0 > 0\} \cap H_e^s = \emptyset \;\&\; \{x|\langle 0,x\rangle \in D_u\} \subseteq K^s \;\& \\ \qquad \Phi_e^s(D_u) \text{ non-single-valued}], \quad \text{if such a } u \text{ exists,} \\ 0, \quad \text{otherwise,} \end{cases}$$

$$m(e,s) = \mu x \cdot [(x)_0 > 0 \;\&\; (\forall i \leq e)(\forall t \leq s)[x > \max D_{u(i,t)}] \\ \;\&\; (\forall i < e)[x > m(i,s) \& x > r(i,s)]],$$

$$r(e,s) = \begin{cases} \mu x \cdot [x \in W_e^s \;\&\; (x)_0 > 0 \;\&\; x \geq m(e,s)], & \text{if such a number exists,} \\ 0, & \text{otherwise.} \end{cases}$$

Then define A^{s+1} by letting $x \in A^{s+1}$ if for some $e \leq s$ one of the following holds:

(a) $(\exists y)[x = \langle 0,y\rangle \,\&\, y \in K^s]$;

(b) $(\exists t \leq s)[x \in D_{u(e,t)} \& (\forall i < e)[(x)_0 > 0 \Rightarrow x \neq r(i,s)]]$;

(c) $r(e,s) = 0 \,\&\, x = m(e,s)$.

This ends Step $(s+1)$.

Now define A by $x \in A$ if $(\exists t)(\forall s \geq t)[x \in A^s]$: since the relation $x \in A^s$ is recursive, A is clearly a Σ_2-set.

SUBLEMMA 1. $(\forall e)[\lim_s u(e,s), \lim_s m(e,s), \lim_s r(e,s)$ *exist*$]$.

PROOF OF SUBLEMMA 1. By induction. Let $e \in \omega$ be given and assume that $(\forall i < e)[\lim_s u(i,s), \lim_s m(i,s), \lim_s r(i,s)$ exist]: for every $i < e$, let $r_i = \lim_s r(i,s), m_i = \lim_s m(i,s)$ and let $H_e = \{x | (\exists i < e)[x = r_i]\}$. It easily follows that

$$\lim_s u(e,s) = \begin{cases} \mu u \cdot [\{x \in D_u | (x)_0 > 0\} \cap H_e = \varnothing \;\&\; \{x | \langle 0, x\rangle \in D_u\} \subseteq K \;\& \\ \qquad \Phi_e(D_u) \text{ non-single-valued}], \quad \text{if such a } u \text{ exists}, \\ 0, \quad \text{otherwise.} \end{cases}$$

Thus, letting $D = \{x | (\exists i \leq e)(\exists t)[x \in D_{u(i,t)}]\}$, since D is finite we have that $\lim_s m(e,s) = ux \cdot [(x)_0 > 0 \& x > \max D \& (\forall i < e)[x > m_i \& x > r_i]]$. The proof of Sublemma 1 is now complete since clearly $\lim_s m(e,s)$ exists $\Rightarrow \lim_s r(e,s)$ exists. Q.E.D.

For every $e \in \omega$, let $m_e = \lim_s m(e,s)$ and $r_e = \lim_s r(e,s)$. Notice that if $r_e \neq 0$ then for every $i > e$, $r_e \in H_i$ and for every $i < e$, $r_e \neq m_i$: hence for cofinitely many s, $r_e \notin A^s$ and thus $r_e \notin A$.

SUBLEMMA 2. $(\forall e)[\Phi_e(A)$ *total* $\Rightarrow \Phi_e(A)$ *recursively enumerable*].

PROOF OF SUBLEMMA 2. Suppose that $\Phi_e(A)$ is total. Let

$$K_0 = \{x | (\exists y \in K)[x = \langle 0, y\rangle]\}$$

and let

$$\bar{H}'_e = \bar{H}_e \cap \{x | (x)_0 > 0\}.$$

We claim that $\Phi_e(A) = \Phi_e(K_0 \cup \bar{H}'_e)$.

Indeed, that $\Phi_e(A)$ is included in $\Phi_e(K_0 \cup \bar{H}'_e)$ is a consequence of the fact that $A \subseteq K_0 \cup \bar{H}'_e$. Suppose now that for some $\langle y, v\rangle \in \omega$, $\langle y, v\rangle \in \Phi_e(K_0 \cup \bar{H}'_e)$ and $\langle y, v\rangle \notin \Phi_e(A)$; since $\Phi_e(A)$ is total, there exists $w \neq v$ such that $\langle y, w\rangle \in \Phi_e(A)$. But, then, for some finite set $D \subseteq A$, we have $\langle y, w\rangle \in \Phi_e(D)$ and therefore there exists a finite set E such that $E \subseteq K_0 \cup \bar{H}'_e$ and $\langle y, v\rangle, \langle y, w\rangle \in \Phi_e(E)$, i.e. $\Phi_e(E)$ is not single-valued. The construction ensures that in this case $\Phi_e(A)$ is not single-valued, contradicting the assumption that $\Phi_e(A)$ is total. We have shown that if $\Phi_e(A)$ is total then $\Phi_e(A) = \Phi_e(K_0 \cup \bar{H}'_e)$ but the latter set is manifestly recursively enumerable and thus the sublemma is proved. Q.E.D.

SUBLEMMA 3. $(\forall e)[A \neq W_e]$.

PROOF OF SUBLEMMA 3. Let $e \in \omega$ be given. We distinguish two cases.

Case 1. $(\exists x)[x \in W_e \,\&\, (x)_0 > 0 \,\&\, x \geq m_e]$. In this case, r_e equals the least such x: thus $r_e \in W_e$ but, as already remarked, $r_e \notin A$ and therefore $A \neq W_e$ as desired.

Case 2. Otherwise. In this case, $(\forall x)[x \in W_e \,\&\, (x)_0 > 0 \Rightarrow x < m_e]$ and, for cofinitely many s, $m_e \in A^s$; thus $m_e \in A - W_e$ and the proof is complete. Q.E.D.

We are now in a position to conclude the proof of the theorem. Indeed, Sublemma 2 and Sublemma 3 ensure that $[A]_e$ is quasi-minimal.

By Sublemma 1 we have that A is a Δ_2-set; moreover, $(\forall y)[y \in K \Leftrightarrow \langle 0, y\rangle \in A]$. Thus $K \leq_1 A$ (which implies $\bar{K} \leq_e \bar{A}$) and, by Lemma (b), also $\bar{A} \leq_e \bar{K}$, since $\bar{A} \in \Delta_2$. Therefore $\bar{A} \equiv_e \bar{K}$ and, by Lemma (a), $[\bar{A}]_e$ is total.

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