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A GENERALIZATION OF THE POINCARÉ-BIRKHOFF THEOREM

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ABSTRACT. We substitute Poincaré's twist hypothesis by the weakest possible topological one: that the homeomorphism in question not be conjugate to a translation.

Let $A = S^1 \times [0,1]$ denote the annulus and $B = R \times [0,1]$ its universal cover; let $T\colon B \to B$ be the translation $T(x,y) = (x + 2\pi, y)$ for $(x,y) \in R \times [0,1]$.

Let $h\colon B \to B$ be a lifting of a homeomorphism $\bar{h}\colon A \to A$ (i.e. $hT = Th$). Recall that h is said to be topologically conjugate to T, if there exists a homeomorphism $k\colon B \to B$ such that $hk = kT$, we write $h \sim T$ if such a k exists, $h \nsim T$ otherwise.

The purpose of this note is to prove the

THEOREM. *Let $\bar{h}\colon A \to A$ be boundary component and orientation preserving; if $h\colon B \to B$ is a lifting of $\bar{h}$ such that $h \nsim T$, then either h has at least one fixed point or there exists in A a closed, simple, noncontractible curve C such that $\bar{h}(C) \cap C = \varnothing$.*

In other words, in the Poincaré-Birkhoff Theorem we substitute Poincaré's twist condition (i.e. that h send the boundary components of B in opposite directions) by the weakest possible condition $h \nsim T$.

Our proof is just a short addendum to Kèrèkjàrtò's proof of the Poincaré-Birkhoff Theorem using Brouwer's translation theory (see [**5**]).

The example in Figure 1 of [**3**] shows that, unlike in the area-preserving case, the existence of only one fixed point is best possible here.

For other generalizations and references see [**3** and **4**].

Proof of the Theorem. We first recall

BROUWER'S LEMMA (SEE [**2**, SATZ 8; **6**, SATZ 9]). *Let $H\colon R^2 \to R^2$ be an orientation-preserving, fixed-point free homeomorphism of the plane. Then, for any point $P \in R^2$, the set $\{H^n(P),\ n \text{ an integer}\}$ has no accumulation point in R^2.*

We call an arc α joining the boundary components of B *free* (with respect to h) if $\alpha \cap h(\alpha) = \varnothing$.

LEMMA. *If h has a free simple arc and $h \nsim T$, then h has a fixed point.*

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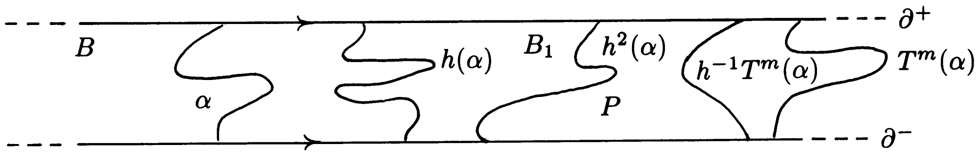

FIGURE 1

PROOF. Assume h has no fixed points and, without loss of generality, let h send both boundary components to the right.

For every $m \geq 0$ let B_m denote the component of $B - h^m(\alpha)$ lying on the right. We claim $\bigcap_{i=0}^{\infty} B_i = \varnothing$.

Suppose $P \in \bigcap_{i=0}^{\infty} B_i$ and let $m \geq 0$ be such that P lies in the left-hand side component of $B - T^m(\alpha)$; since $hT = Th$, $h^{-1}(T^m(\alpha))$ lies in that component also (see the figure). Since the sequence $h^{-n}(P)$, $n \geq 0$, lies entirely in the compact region of B bounded by α and $T^m(\alpha)$, we have found a contradiction to Brouwer's Lemma.

Now it is easy to see directly that the orbit space B/h is Hausdorff and the natural projection a covering map i.e. B/h is homeomorphic to the cylinder $S^1 \times [0,1]$ (this is a special case of "Sperner's criterion" (see [**6**, Satz 27] or [**1**, p. 73]).

Hence, if $k\colon B \to B$ is a lifting of a homeomorphism $\bar{k} : B/h \to S^1 \times [0,1]$ we have $hk = kT$ i.e. $h \sim T$, and the Lemma is proven.

To prove the Theorem simply observe that in his proof of the Poincaré-Birkhoff Theorem, Kèrèkjàrtò constructs a simple, topological halfline L, such that $L \cap h(L) = \varnothing$, starting from one boundary component ∂^+ of B, and uses Poincaré's twist condition *only* to conclude that L cannot cross the other boundary component ∂^-; see p. 101 of [**5**]. (This fact then allows the construction of the closed curve C.)

However, if the line L *does* intersect the boundary component ∂^-, then we have obtained a free arc α for h and the existence of the fixed point follows from our Lemma.

A conjecture. Unlike Poincaré's twist condition, the condition $h \not\sim T$ still makes sense when one or both boundary components of the annulus A shrink to a point, leading us to venture a conjecture.

Let S^2 denote the two-dimensional sphere and let $\bar{h} : S^2 \to S^2$ be an orientation and *area*-preserving homeomorphism with two distinct *stable* fixed points N and S; consider the plane R^2 as the universal cover of $S^2 - N \cup S$;

CONJECTURE. If $h : R^2 \to R^2$ is a lifting of $\bar{h}$ and $h \not\sim T$, then h has a fixed point.

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