A GENERALIZATION OF
THE POINCARÉ-BIRKHOFF THEOREM

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ABSTRACT. We substitute Poincaré's twist hypothesis by the weakest possible topological one: that the homeomorphism in question not be conjugate to a translation.

Let \( A = S^1 \times [0,1] \) denote the annulus and \( B = \mathbb{R} \times [0,1] \) its universal cover; let \( T : B \rightarrow B \) be the translation \( T(x,y) = (x + 2\pi, y) \) for \((x,y) \in \mathbb{R} \times [0,1] \).

Let \( h : B \rightarrow B \) be a lifting of a homeomorphism \( \bar{h} : A \rightarrow A \) (i.e. \( hT = \bar{h}h \)). Recall that \( h \) is said to be topologically conjugate to \( T \), if there exists a homeomorphism \( k : B \rightarrow B \) such that \( hk = kT \), we write \( h \sim T \) if such a \( k \) exists, \( h \not\sim T \) otherwise.

The purpose of this note is to prove the

**Theorem.** Let \( \bar{h} : A \rightarrow A \) be boundary component and orientation preserving; if \( h : B \rightarrow B \) is a lifting of \( \bar{h} \) such that \( h \not\sim T \), then either \( h \) has at least one fixed point or there exists in \( A \) a closed, simple, noncontractible curve \( C \) such that \( \bar{h}(C) \cap C = \emptyset \).

In other words, in the Poincaré-Birkhoff Theorem we substitute Poincaré’s twist condition (i.e. that \( h \) send the boundary components of \( B \) in opposite directions) by the weakest possible condition \( h \not\sim T \).

Our proof is just a short addendum to Kerekjarto’s proof of the Poincaré-Birkhoff Theorem using Brouwer’s translation theory (see [5]).

The example in Figure 1 of [3] shows that, unlike in the area-preserving case, the existence of only one fixed point is best possible here.

For other generalizations and references see [3 and 4].

**Proof of the Theorem.** We first recall

**Brouwer’s Lemma** (see [2, Satz 8; 6, Satz 9]). Let \( H : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be an orientation-preserving, fixed-point free homeomorphism of the plane. Then, for any point \( P \in \mathbb{R}^2 \), the set \( \{H^n(P), n \text{ an integer} \} \) has no accumulation point in \( \mathbb{R}^2 \).

We call an arc \( \alpha \) joining the boundary components of \( B \) free (with respect to \( h \)) if \( \alpha \cap h(\alpha) = \emptyset \).

**Lemma.** If \( h \) has a free simple arc and \( h \not\sim T \), then \( h \) has a fixed point.
PROOF. Assume $h$ has no fixed points and, without loss of generality, let $h$ send both boundary components to the right.

For every $m \geq 0$ let $B_m$ denote the component of $B - h^m(\alpha)$ lying on the right. We claim $\bigcap_{i=0}^{\infty} B_i = \emptyset$.

Suppose $P \in \bigcap_{i=0}^{\infty} B_i$ and let $m \geq 0$ be such that $P$ lies in the left-hand side component of $B - T^m(\alpha)$; since $hT = Th$, $h^{-1}(T^m(\alpha))$ lies in that component also (see the figure). Since the sequence $h^{-n}(P)$, $n \geq 0$, lies entirely in the compact region of $B$ bounded by $\alpha$ and $T^m(\alpha)$, we have found a contradiction to Brouwer’s Lemma.

Now it is easy to see directly that the orbit space $B/h$ is Hausdorff and the natural projection a covering map i.e. $B/h$ is homeomorphic to the cylinder $S^1 \times [0,1]$ (this is a special case of “Sperner’s criterion” (see [6, Satz 27] or [1, p. 73]).

Hence, if $k: B \to B$ is a lifting of a homeomorphism $\bar{k}: B/h \to S^1 \times [0,1]$ we have $hk = kT$ i.e. $h \sim T$, and the Lemma is proven.

To prove the Theorem simply observe that in his proof of the Poincaré-Birkhoff Theorem, Kérèkjhátó constructs a simple, topological halfline $L$, such that $L \subset h(L) = \emptyset$, starting from one boundary component $\partial^+$ of $B$, and uses Poincaré’s twist condition only to conclude that $L$ cannot cross the other boundary component $\partial^-$; see p. 101 of [5]. (This fact then allows the construction of the closed curve $C$.)

However, if the line $L$ does intersect the boundary component $\partial^-$, then we have obtained a free arc $\alpha$ for $h$ and the existence of the fixed point follows from our Lemma.

A conjecture. Unlike Poincaré’s twist condition, the condition $h \not\sim T$ still makes sense when one or both boundary components of the annulus $A$ shrink to a point, leading us to venture a conjecture.

Let $S^2$ denote the two-dimensional sphere and let $\bar{h}: S^2 \to S^2$ be an orientation and area-preserving homeomorphism with two distinct stable fixed points $N$ and $S$; consider the plane $R^2$ as the universal cover of $S^2 - N \cup S$.

CONJECTURE. If $h: R^2 \to R^2$ is a lifting of $\bar{h}$ and $h \not\sim T$, then $h$ has a fixed point.

REFERENCES


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