

BIFURCATION TO BADLY ORDERED ORBITS IN ONE-PARAMETER FAMILIES OF CIRCLE MAPS OR ANGELS FALLEN FROM THE DEVIL'S STAIRCASE

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ABSTRACT. We discuss the structure of the bifurcation set of a one-parameter family of endomorphisms of S^1 having two critical points and negative Schwarzian derivative. We concentrate on the case in which one of the endpoints of the rotation set is rational, providing a partial characterization of components of the nonwandering set having specified rotation number and the bifurcations in which they are created. In particular we find, for each rational rotation number p'/q' less than the upper boundary of the rotation set p/q , infinitely many saddle-node bifurcations to badly ordered periodic orbits of rotation number p'/q' .

1. Introduction. In this paper we use the properties of symbol sequences and rotations on $S^1 = \mathbf{R}/\mathbf{Z}$, together with the kneading theory of endomorphisms of S^1 , to obtain information about the structure of the bifurcation set of a one-parameter family of endomorphisms of the circle. The present paper is part of our continuing work on bifurcations of maps of the circle and the annulus (Hockett and Holmes [1986a, b]) and we suspect that the tools used are of general interest. Two important notions in the study of maps of the circle are the rotation number of a point and the rotation set of the map.

DEFINITIONS 1.1. Let $f: S^1 \rightarrow S^1$ be a continuous map of degree 1 and let $\tilde{f}: \mathbf{R} \rightarrow \mathbf{R}$ be a lift of f . The *rotation number* of a point $x \in S^1$ under f is

$$\rho(x; f) = \lim_{n \rightarrow \infty} (\tilde{f}(\tilde{x}) - \tilde{x})/n,$$

if the limit exists. (Here $\tilde{x} \in \mathbf{R}$ is a point which covers $x \in S^1$. That f has degree 1 means $\tilde{f}(\tilde{x} + 1) = \tilde{f}(\tilde{x}) + 1$ for any lift \tilde{f} and $\tilde{x} \in \mathbf{R}$.) The *rotation set* $\rho(f)$ is

$$\rho(f) = \{\rho(x; f) | x \in S^1\},$$

i.e., the set of all rotation numbers $\rho(x; f)$ for f .

It is well known (see, e.g., Coddington and Levinson [1955]) that for homeomorphisms of the circle all points $x \in S^1$ rotate with the same asymptotic rate, so that $\rho(f)$ consists of a single point. For noninvertible maps, however, this need not be the case. In general $\rho(f)$ is known to be a closed interval, possibly degenerating to a point (Ito [1981], Newhouse et al. [1983]). Letting $\rho(f) = [\rho^-(f), \rho^+(f)]$, we call $\rho^-(f)$ and $\rho^+(f)$ the *upper* and *lower rotation bounds* of f , respectively. Since we are interested in one-parameter families of circle endomorphisms f_μ we will also

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use the notations $\rho(x; \mu)$, $\rho(\mu)$, $\rho^-(\mu)$ and $\rho^+(\mu)$ for the rotation number, rotation set and rotation bounds of f_μ , respectively.

The precise family of maps that we consider is defined in §2.

For now, note that we require 0 to be an unstable fixed point for each member f_μ of our one-parameter family: $f_\mu(0) = 0$, $f'_\mu(0) > 1$. This, together with assumption (M1) of §2, implies that $\rho^-(\mu) = 0$ for all μ . This fact simplifies the discussion of a number of our results, but we expect similar results to hold for more general classes of maps.

Recall that the *orbit* of a point $x \in S^1$ is the set $\mathcal{O}(x) = \{f^i(x)\}_{i=0}^\infty$ and that x is *q-periodic* if there exists $q \geq 0$ such that $f^q(x) = x$ and $f^i(x) \neq x$ for $i = 1, \dots, q-1$. Periodic points always possess a (rational) rotation number.

An important aspect of the dynamics of a point $x \in S^1$ is the order structure of the orbit $\mathcal{O}(x)$.

DEFINITION 1.2. The orbit of a point $x \in S^1$ under f is said to be *well ordered* or *order preserving* if for any lift \bar{f} of f to \mathbf{R} and *any* two points $y, y' \in \mathcal{O}(x)$ such that $0 < y < y'$ (with respect to the cyclic ordering on S^1) we have $\bar{f}(\bar{y}) < \bar{f}(\bar{y}')$, where \bar{y}, \bar{y}' cover y, y' , respectively, and $\bar{y}' - \bar{y} < 1$. Equivalently, if $y, y', y'' \in \mathcal{O}(x)$ are any three points in the orbit of x and $y < y' < y''$ with respect to the cyclic ordering on S^1 , then $f(y) < f(y') < f(y'')$ with respect to that ordering. Points not satisfying the above are called *badly ordered*.

For the families of maps studied here, $\rho^+(\mu)$ is a Cantor function or ‘Devil’s staircase’ which takes rational values on intervals and irrational values at points (Boyland [1986]). See Figure 2(b). Let $[s(p/q), e(p/q)]$ be a rational interval (or, more picturesquely, let $[s(p/q), e(p/q)] \times \{p/q\}$ be a rational step of the staircase). One of our main results, stated somewhat imprecisely, is the following: for any $p'/q' \leq p/q$ there exist infinitely many parameter values $\mu \in (s(p/q), e(p/q))$ at which saddle-node bifurcations to periodic orbits of rotation number p'/q' occur. Every such periodic orbit, and indeed every periodic orbit which bifurcates from it, is badly ordered. For $p'/q' < p/q$ one may, in some sense, regard these badly ordered periodic orbits as being created “below the staircase” (Figure 2(c)), hence our subtitle.

In what follows we study the bifurcations which occur in a one-parameter family of maps f_μ as μ varies over a rational step $[s(p/q), e(p/q)]$. (The bifurcation occurring at an irrational point is examined in Hockett and Holmes [1986b]). Similar questions have been addressed by various authors, notably Boyland [1986], Bernhard [1982], Block [1973, 1980, 1981], Block and Franke [1973], Block et al. [1980], Chenciner et al. [1984a, b], Gambaudo et al. [1984], MacKay and Tresser [1984, 1985b], Misiurewicz [1984], Newhouse et al. [1983], Kadanoff [1983] and Hockett and Holmes [1986b]. With the exception of Boyland, Misiurewicz and Hockett-Holmes, these works have generally focused on those bifurcations which result in changes to the rotation set $\rho(\mu)$, in particular saddle-node bifurcations to well ordered periodic orbits. In this paper, in contrast, we study the bifurcations occurring while $\rho^+(\mu)$ remains rational and constant.

Finally, we remark that our results extend to dissipative maps of annuli and can be connected to those of Aubry [1983], Aubry and Le Daeron [1983], Katok [1982, 1983] Mather [1982a, b, 1984] and Boyland and Hall [1987] on area-preserving

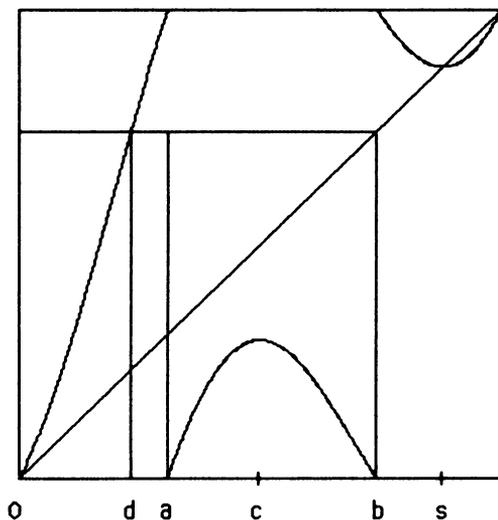


FIGURE 1. A map $f_\mu \in \mathcal{M}$

twist maps. We also note the related results of Casdagli [1985] and Le Calvez [1985] on the structure of ‘Birkhoff attractors’ (Birkhoff [1932]).

2. The class of maps \mathcal{M} . For simplicity we consider the following class of maps, although our methods should generalize, at the possible expense of more complicated kneading theory (see, e.g., MacKay and Tresser [1985a]). Let $f_\mu: S^1 \rightarrow S^1$ denote a member of a one-parameter family \mathcal{M} of endomorphisms of $S^1 = \mathbf{R}/\mathbf{Z}$ such that

(M1) f_μ is C^3 and has two critical points $0 < c < s < 1$ with $f''_\mu(c) < 0 < f''_\mu(s)$, and s lies in the immediate basin of an attractive fixed point w , i.e., s lies in the connected component of the basin of w which contains w .

(M2) f_μ has negative Schwarzian derivative:

$$S(f_\mu) = f'''_\mu / f'_\mu - \frac{3}{2} (f''_\mu / f'_\mu)^2 < 0 \quad \text{on } S^1 - \{c, s\}.$$

(M3) $0 = 1$ is an unstable fixed point for f_μ , and $f'_\mu(0) > 1$.

(M4) There exist parameter values μ_0, μ_1 such that for $\mu \in (\mu_0, \mu_1)$ there are points $a, b \in S^1$ with $f_\mu(a) = f_\mu(b) = 0$ and $0 < a < c < b < s$; $\mu_0 = \inf\{\mu \mid f_\mu(c) = 0\}$, for $\mu \in (\mu_0, \mu_1)$ we have $f_\mu(c) < b$ and $f_{\mu_1}(c) = b$. See Figure 1.

Note, in particular, that (M2) implies that a version of *Singer’s theorem* (Singer [1978], Misiurewicz [1981a]) holds for maps in \mathcal{M} : if f_μ has a stable periodic orbit in $[0, b]$ then the critical point c lies in its basin of attraction. Note also that for $\mu \leq \mu_0$, $\rho(f_\mu) = \{0\}$ while for $\mu \geq \mu_1$, $\rho(f_\mu) = [0, 1]$.

In terms of the usual two-parameter families of circle maps, the ‘canonical’ example of which is $F_{\delta, \varepsilon}(x) = x + \varepsilon + (\delta/2\pi) \sin(2\pi x)$, our family traverses a path $(\delta(\mu), \varepsilon(\mu))$ which lies inside and near the right-hand boundary of the 0/1 ‘Arnold tongue’ (Boyland [1986], MacKay and Tresser [1985]). We note that the requirement in (M1) that the critical point s lie in the immediate basin of a stable fixed point can be relaxed somewhat by requiring, instead, that $s \in (b, 1)$

and $f_\mu([b, 1]) \subset [b, 1]$, i.e., $x = 1$ is a *central restrictive point* for f_μ (Guckenheimer [1979], Guckenheimer and Holmes [1983]). Since our arguments do not depend on the details of the dynamics of f_μ in $[b, 1]$ and since in the canonical two-parameter family $F_{\delta, \varepsilon}$ it is easily arranged that our one-parameter family $f_\mu = F_{\delta(\mu), \varepsilon(\mu)}$ have a stable fixed point (Boyland [1986]), there is no particular gain in generality in weakening (M1). A family of maps which satisfies (M1)–(M4), and indeed possesses a superstable fixed point for all nonnegative values of the parameter μ , is

$$(2.1) \quad x \mapsto f_\mu(x) = x + \frac{\mu}{2\pi} + \frac{1}{2\pi}[\sin(2\pi x) - \mu \cos(2\pi x)]; \quad \mu \geq 0.$$

It simplifies the statement of some of our results if we further assume that \mathcal{M} is a *regular, versal* family. Regularity implies that the only bifurcations of periodic points of f_μ are saddle-node and period doubling bifurcations. Versality implies, roughly speaking, that the topological entropy of f_μ increases monotonically with μ and, moreover, that *all* bifurcation points (periodic, nonperiodic, homoclinic, etc.) and their accumulation points are passed with nonzero speed (cf. Jonker and Rand [1981a, b]).

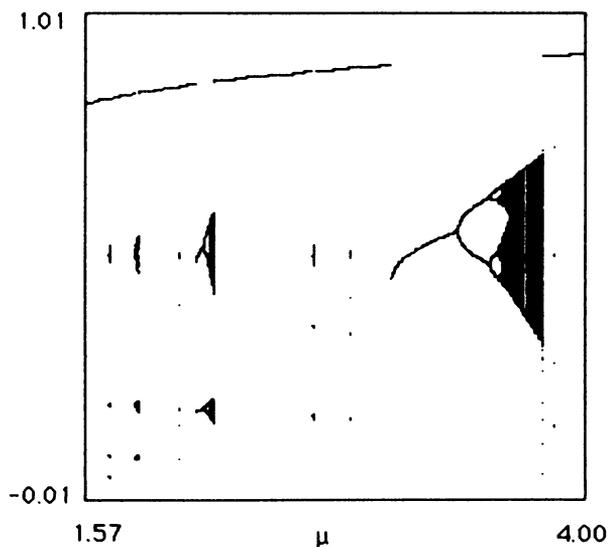


FIGURE 2(a). The attracting sets for the family f_μ of equation (2.1)

Since Singer's theorem holds for the family f_μ of (2.1), we can perform the following numerical experiment (Collet and Eckmann [1980]). For each of 1000 values of $\mu \in (1.57, 4.00)$ we iterate the critical point $c = \frac{1}{2} 425$ times, plotting the last 125 iterates over μ . The result of the experiment is shown in Figure 2(a). As μ varies from 1.57 to 4.00, we observe that the critical point is alternately attracted to the superstable fixed point s (the 'curve' at the top of the figure) and attracting sets which are reminiscent of the behavior one observes in one-parameter families of quadratic maps. The most visible examples of the latter kind of behavior are stable periodic orbits of periods 1 and 2 and the attracting sets which bifurcate

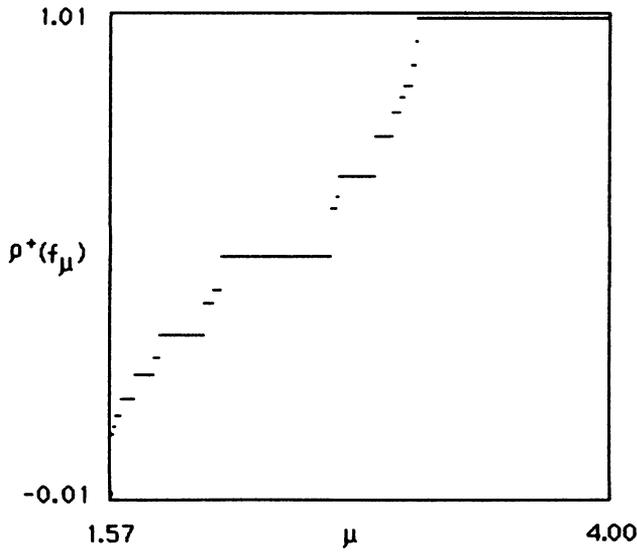


FIGURE 2(b). $\rho^+(f_\mu)$ as a function of μ

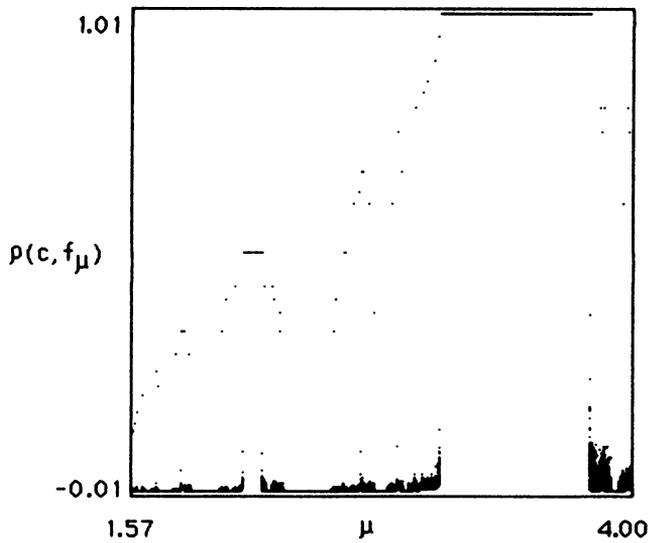


FIGURE 2(c). $\rho(c, f_\mu)$ as a function of μ

from them. Also visible are two period 3 ‘structures’ which, as we shall see, can be distinguished by their rotation numbers, one being $\frac{1}{3}$, the other $\frac{2}{3}$.

In Figure 2(b) we plot $\rho^+(\mu)$ as a function of μ for the family (2.1), $\mu \in (1.57, 4.00)$; ρ^+ was obtained numerically by calculating $\rho(1/2; \bar{\mu})$ for 50,000 values of $\bar{\mu} \in (1.57, 4.00)$ and then plotting $\max_{\mu \leq \bar{\mu}} \rho(1/2; \mu)$. Figure 2(c) shows $\rho(1/2; \mu)$, the rotation number of the critical point, as a function of μ . Figures 2(d) and (e) show blowups of Figures 2(a) and (c) for the case $\rho^+(\mu) = 1/2$. We note four

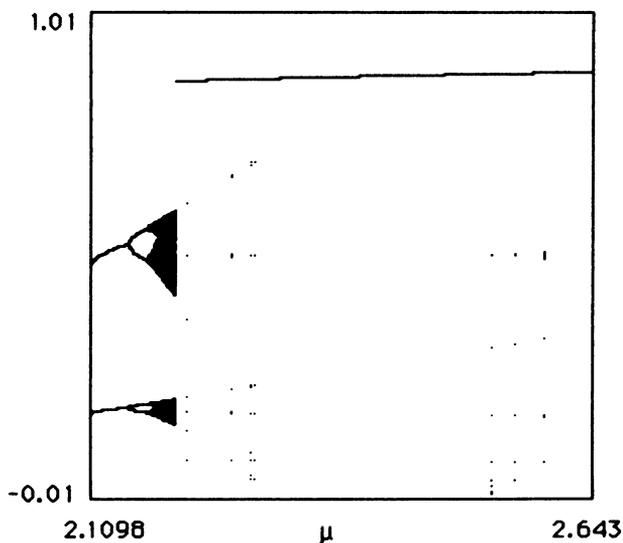


FIGURE 2(d). The attracting sets of f_μ on the $1/2$ -step

aspects of these numerical results. (1) The upper rotation bound $\rho^+(\mu)$ is a Cantor function which takes rational values on intervals (Boyland [1986]). (2) At the beginning of each such 'rational step' with $\rho^+(\mu) = p/q$, a saddle-node bifurcation occurs in which a pair of q -periodic orbits appear (Boyland [1986]). (3) The p/q -step persists beyond the value of μ at which the ' q -periodic family' emerging from this saddle-node loses stability. Thus the upper rotation bound $\rho^+(\mu)$ need not be associated with an attractor. (4) After the q -periodic family loses stability there are evidently new stable periodic orbits created with rotation numbers $\leq p/q$ before any orbits with $\rho > p/q$ appear (Figures 2(d),(e)). The theory that follows will explain these observations.

DEFINITION 2.1. Let $f_\mu \in \mathcal{M}$, $p/q \in [0, 1]$ with p, q relatively prime positive integers and $\alpha \in [0, 1] - \mathcal{Q}$. We define

$$s(p/q) = \inf\{\mu \geq \mu_0 \mid \rho^+(\mu) = p/q\},$$

$$e(p/q) = \sup\{\mu \geq \mu_0 \mid \rho^+(\mu) = p/q\}$$

and

$$i(\alpha) = \inf\{\mu \geq \mu_0 \mid \rho^+(\mu) = \alpha\}.$$

One can characterize $s(p/q)$ and $i(\alpha)$ dynamically as follows: $s(p/q)$ is a bifurcation point at which a saddle-node bifurcation occurs in which a pair of *well ordered* q -periodic orbits of rotation number p/q appear (Boyland [1986], Hockett [1986]). The point $i(\alpha)$ is a bifurcation point at which a *well ordered* Cantor set of rotation number α is created, i.e., an invariant Cantor set, each point of which has rotation number α and a well ordered orbit, appears (Boyland [1986]). Moreover, the pair of q -periodic orbits created at $s(p/q)$ and the Cantor set created at $i(\alpha)$ are the *only* invariant sets of f_μ ($\mu > s(p/q)$ and $\mu > i(\alpha)$, respectively) of rotation number p/q and α , respectively, which are well ordered (Hockett [1986]).

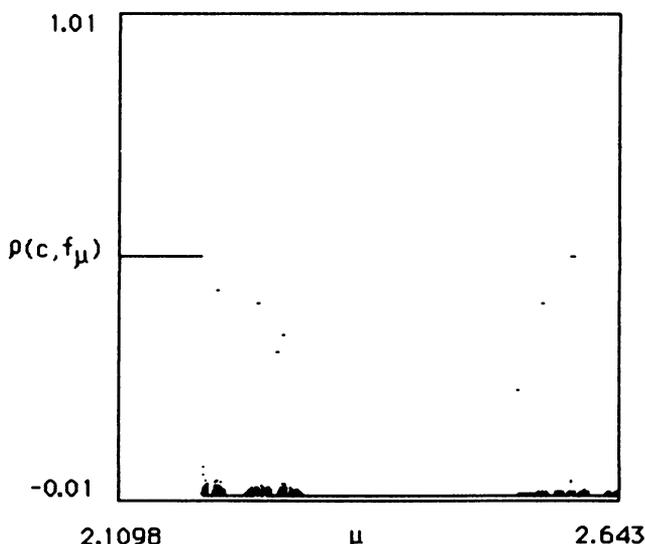


FIGURE 2(e). $\rho(c, f_\mu)$ on the 1/2-step

3. Kneading theory for maps in \mathcal{M} . In this section we very briefly review the kneading theory of Milnor and Thurston [1977] as applied to endomorphisms of S^1 . For a more complete treatment see Bernhardt [1982], Collet and Eckmann [1980], Devaney [1986], MacKay and Tresser [1985b] and especially Guckenheimer [1977, 1979, 1980].

Let f_μ denote a one-parameter family of maps in \mathcal{M} and let $d = f_\mu^{-1}(b)$. Note that d is unique since f_μ^{-1} is 1-1 on $(f_\mu(c), s)$. It is easily seen that $0 = s_0 < d < a$ (Figure 3). We now define $g_\mu : [0, d] \cup [a, b] \rightarrow [0, b]$ to be the restriction of f_μ to $[0, d] \cup [a, b]$. Hence we forget about the action of f_μ on the 'gap' $G = (d, a)$ (see Figure 3). Notice that $f_\mu([d, a]) = f_\mu(Cl(G)) = [b, 1]$ and $f_\mu([b, 1]) \subset [b, 1]$ (so that $[b, 1]$ is a restrictive interval). Rephrased, all points in $Cl(G) = [d, a]$ are wandering for f_μ and the nonwandering set of f_μ is contained in $[0, d] \cup [a, b] \cup \{s\}$.

Let $I = [0, b]$ and for $x \in I$ let

$$(3.1) \quad \varepsilon_n(x) = \begin{cases} 0 & \text{if } g^n(x) \in I_0 = [0, d], \\ G & \text{if } g^n(x) \in G = (d, a), \\ 1 & \text{if } g^n(x) \in I_1 = [a, c], \\ C & \text{if } g^n(x) = c, \\ 2 & \text{if } g^n(x) \in I_2 = (c, b]. \end{cases}$$

See Figure 3. In what follows we shall frequently write g in place of g_μ when reference to the parameter is unimportant. We denote by $\varepsilon(x)$ the sequence $\{\varepsilon_n(x)\}_{n=0}^\infty$ and adopt the convention that $\varepsilon_k(x) = G$ implies $\varepsilon_j(x) = G$ for all $j > k$. This sequence is called the *itinerary* of x .

Let \mathbf{a} and \mathbf{b} be two sequences of the symbols $\{0, G, 1, C, 2\}$ such that $\mathbf{a} \neq \mathbf{b}$. Suppose i is the smallest index for which $a_i \neq b_i$ and $a_i < b_i$ (in the lexicographic order $0 < G < 1 < C < 2$). If the leading part of the sequences $a_0 a_1 \cdots a_{i-1} = b_0 b_1 \cdots b_{i-1}$ contain an *even* number of 2's then we order \mathbf{a} and \mathbf{b} lexicographically,

Conversely let \mathbf{a} be a sequence of the symbols $\{0, G, 1, C, 2\}$ satisfying

$$\begin{aligned} a_k = 0 &\Rightarrow \nu_2 < \sigma^{k+1}(\mathbf{a}) < \nu_1, \\ a_k = 1 \text{ or } 2 &\Rightarrow \nu_2 < \sigma^{k+1}(\mathbf{a}) < \nu_3, \\ a_k = C &\Rightarrow \sigma^k(\mathbf{a}) = \varepsilon(c). \end{aligned}$$

Then there exists $x \in I$ with $\varepsilon(x) = \mathbf{a}$.

Note that for the families of maps under consideration, the only kneading sequence that changes with μ is $\nu_3 = \varepsilon(g_\mu(c))$. Hence changes in the structure of the nonwandering set of g_μ are controlled by the behavior of the critical point c .

We shall also need the following:

PROPOSITION 3.4. *Let $x \in I - G$. Then*

$$\rho(x; f_\mu) = R(\varepsilon(x)) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \text{rot}(\varepsilon_i(x))$$

where $\text{rot}(\varepsilon_i) = 0$ if $\varepsilon_i \in \{0, G\}$ and $\text{rot}(\varepsilon_i) = 1$ otherwise.

The proof is entirely elementary and simply makes use of the fact that points $x \in I_0 \cup G$ fail to make an entire trip around S^1 on applying f_μ , whereas points $x \in I_1 \cup \{c\} \cup I_2$ do make a full trip around S^1 when f_μ is applied to them. For details see Hockett and Holmes [1986a, b].

In what follows we use the above results to make symbolic calculations which, in turn, yield the structure of the bifurcation set on a rational step.

4. The main theorem. Recall that at $s(p/q)$ a saddle-node bifurcation occurs in which a pair of well ordered q -periodic orbits of rotation number p/q is created. this implies that at $\mu = s(p/q)$ the map g_μ^q acquires q fixed points $\{t_0, \dots, t_{q-1}\}$, Precisely p of which lie in I_1 . Now for each point t_i there exists a point t'_i such that $g_\mu^q(t_i) = g_\mu^q(t'_i)$ for $\mu \geq s(p/q)$ (see Figure 4), so that the points t_i are *central restrictive points* for g_μ (Guckenheimer [1979], Guckenheimer and Holmes [1983, §5.6]). That is, each interval $[t_i, t'_i]$ contains a point c_i such that $(g_\mu^q)'(c_i) = 0$, g_μ^q is monotone on $[t_i, c_i]$ and $g_\mu^q([t_i, t'_i]) \subset [t_i, t'_i]$. The existence of the points c_i is guaranteed by Singer's theorem (see Singer [1978] or especially Misiurewicz [1981a]). We will let $c_0 = c$ so that t_0 is the point of the periodic orbit $\{t_i\}_{i=0}^{q-1}$ closest to c , and we index so that $f_\mu(t_i) = t_{(i+1) \bmod q}$.

Next observe that $g_\mu^q(c)$ increases monotonically with μ for $\mu \geq s(p/q)$. Thus we have the following.

DEFINITION 4.1. Let $f_\mu \in \mathcal{M}$. Then we define

$$\begin{aligned} h(p/q) &= \inf\{\mu > s(p/q) \mid f_\mu^q(c) = t'_0\}, \\ g(p/q) &= \inf\{\mu > h(p/q) \mid f_\mu^q(c) = b\} \end{aligned}$$

and

$$g'(p/q) = \inf\{\mu > g(p/q) \mid f_\mu^q(c) = 1 (= 0)\}.$$

Notice that for $\mu = h(p/q)$ the maps $G_{\mu,i} = g_\mu^q|_{[t_i, t'_i]}$ are onto, and hence for $\mu > h(p/q)$ the map g_μ loses the restrictive intervals $[t_i, t'_i]$ created in the saddle-node bifurcation at $s(p/q)$. (This remark uses the versality hypothesis of §2.) Also,

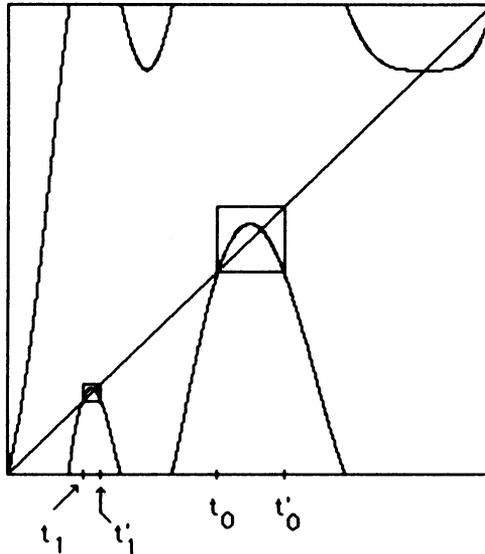


FIGURE 4. The map f_μ^2 for $\mu \in (s(1/2), h(1/2))$

for $\mu \in (g(p/q), g'(p/q))$ notice that $g_\mu^{q-1}(c) \in G$. We shall refer to the interval $[g(p/q), g'(p/q)]$ as the *fat gap* and whenever $\mu \in [s(p/q), e(p/q)]$ is such that $f_\mu^k(c) \in G$ for some $k > 0$ we shall say that the critical point has fallen into a *gap*.

We can now state our main result.

THEOREM 4.2. *Let $f_\mu \in \mathcal{M}$ and let $\mu \in [s(p/q), e(p/q)]$. Then we have the following.*

(i) *For $\mu \in [s(p/q), h(p/q)]$, $\rho(c; \mu) = p/q$ and the nonwandering set of f_μ has a decomposition $\Omega(\mu) = Q(\mu) \cup \mathcal{S} \cup \{s\}$ where \mathcal{S} is topologically equivalent to a subshift of finite type on $q + 2$ symbols and $Q(\mu)$ is the union of the nonwandering sets $\Omega(G_{\mu,i})$ for $i = 0, \dots, q - 1$. Each $\Omega(G_{\mu,i})$ has a ‘canonical’ decomposition. (Here $G_{\mu,i} = g_\mu^q[[t_i, t'_i]] = f_\mu^q[[t_i, t'_i]]$.)*

(ii) *There exist sequences $\{g_i(p/q)\}$ and $\{g'_i(p/q)\}$ (the index i being determined by p and q) with $g_i(p/q), g'_i(p/q) < g(p/q)$ such that $\mu = g_i(p/q)$ implies $f_\mu^i(c) = a$, $\mu = g'_i(p/q)$ implies $f_\mu^i(c) = d$ and $\mu \in (g_i(p/q), g'_i(p/q))$ implies $f_\mu^i(c) \in G$. The point $h(p/q)$ is accumulated on by the sequences $\{g_i(p/q)\}$ and $\{g'_i(p/q)\}$.*

(iii) *Consider the interval $(g'_{i_{k+1}}(p/q), g_{i_k}(p/q))$. Then*

(a) *for any $p'/q' < p/q$ there is a sequence $\{s'_i(p'/q')\}$ of parameter values with $s'_i(p'/q') \rightarrow g'_{i_{k+1}}(p/q)$ from above as $i \rightarrow \infty$ at which saddle-node bifurcations to badly ordered periodic orbits of rotation number p'/q' occur;*

(b) *there exists a sequence $\{g'_{i_{k+1},j}\}$ of parameter values with $g'_{i_{k+1},j} \rightarrow g'_{i_{k+1}}(p/q)$ from above as $j \rightarrow \infty$ such that $f_\mu^l(c) \in \partial G$ for some l when $\mu = g'_{i_{k+1},j}$;*

(c) *for any $p'/q' < p/q$ there is a sequence $\{s_i(p'/q')\}$ of parameter values with $s_i(p'/q') \rightarrow g_{i_k}(p/q)$ from below as $i \rightarrow \infty$ at which saddle-node bifurcations to badly ordered periodic orbits of rotation number p'/q' occur;*

(d) there exists a sequence $\{g_{i_{k,j}}\}$ of parameter values with $g_{i_{k,j}} \rightarrow g_{i_k}(p/q)$ from below as $i \rightarrow \infty$ such that $f_\mu^l(c) \in \partial G$ for some l when $\mu = g_{i_{k,j}}$.

(iv) Consider the interval $(g'(p/q), e(p/q))$. Statements similar to (iii)(a)–(d) hold in this interval also, and in addition we have the following. There exists a sequence $\{s_l(p/q)\}$ of parameter values such that $s_l(p/q) \rightarrow e(p/q)$ from below as $l \rightarrow \infty$ at which saddle-node bifurcations to badly ordered periodic orbits of period $q(l + 1)$ and rotation number p/q occur. Moreover, $e(p/q)$ is characterized dynamically by

$$e(p/q) = \inf\{\mu > g'(p/q) \mid f_\mu^q(c) = t_k\}$$

where t_k is the smallest point in the unstable periodic orbit $\{t_i\}_{i=0}^{q-1}$ created at $\mu = s(p/q)$. (Here 'smallest' means $0 < t_k < t_i$ for $i \neq k$ in the cyclic ordering on S^1 .)

(v) Whenever μ is such that $f_\mu^l(c) \in G$ for some l , the nonwandering set of the restricted map $g_\mu = f_\mu|([0, b] - G)$ is topologically equivalent to a subshift of finite type on $l + 2$ symbols.

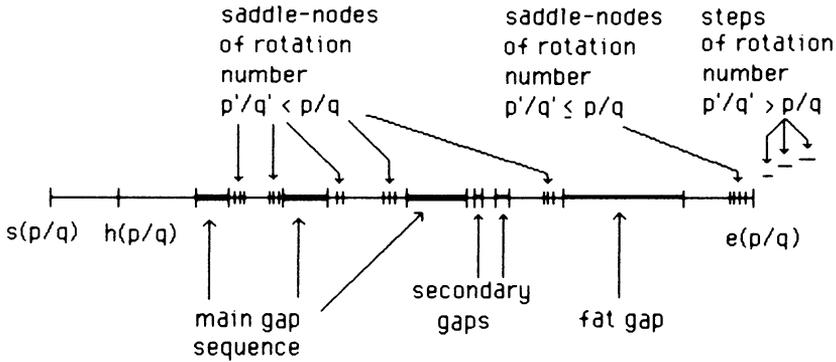


FIGURE 5. The structure of a p/q -step

Figure 5 depicts the structure of the bifurcation set on a rational step as described in Theorem 4.2. We can summarize Theorem 4.2 and some of our remarks preceding the theorem as follows. A p/q -step opens with a saddle-node bifurcation at a parameter value $\mu = s(p/q)$ in which a pair of well ordered q -periodic orbits is created. This leads to a cascade of period-doubling bifurcations and, indeed, there is an interval $[s(p/q), h(p/q)]$ in parameter space on which the nonwandering set of f_μ decomposes into a quadratic-like piece and a subshift of finite type. The point $h(p/q)$ is accumulated on from above by a sequence of 'gaps'. If μ lies in one of these gaps then the critical point c lies in the basin of attraction of the stable fixed point and the nonwandering set of $f_\mu|([0, b] - G)$ is equivalent to a subshift of finite type. The endpoints of these gaps are accumulation points of saddle-node bifurcations to badly ordered periodic orbits of any rotation number smaller than p/q . In addition there are other gaps accumulating on the ends of this primary sequence of gaps. The ends of the 'fat gap' $(g(p/q), g'(p/q))$ are also accumulated on by such saddle-node points and secondary gaps. The p/q -step ends at a value $\mu = e(p/q)$ which is accumulated on from below by saddle-node bifurcation points at which periodic

orbits of rotation number $(l + 1)p/(l + 1)q = p/q$ are created, and from above by saddle-node bifurcations to periodic orbits or rotation number greater than p/q .

REMARKS. (1) We shall actually prove more than is stated in the theorem. Indeed, the method of proof involves symbolically constructing all of the desired periodic orbits, gaps, etc., and the procedure can be applied iteratively to show that there are additional saddle-node bifurcations and gaps accumulating on the secondary gaps of the theorem, and so on. The fallen angels proliferate without end. Thus our methods yield topologically self-similar bifurcation sets similar to the boîtes embôitées observed by Gumowski and Mira [1980] for unimodal maps. We have no doubt that the bifurcation set is rife with metric self similarites, scaling functions and universal numbers.

(2) As pointed out in the Introduction, parts of Theorem 4.2 extend directly to (strongly) dissipative diffeomorphisms of the annulus. In particular, for the family $(\theta, r) \xrightarrow{F_{\mu,\epsilon}} (\theta + \epsilon r + f_\mu(\theta), \epsilon r + f_\mu(\theta))$ having $\det(DF_{\mu,\epsilon}) = \epsilon$, if f_μ has a hyperbolic subshift $S(\mu)$ on l symbols then $F_{\mu,\epsilon}$ has a conjugate subshift for $|\epsilon|$ sufficiently small (depending, in general, on l). As in Holmes-Whitley [1984], saddle-node bifurcation curves extend from the points $s_i(p'/q'), s'_i(p', q')$ into (μ, ϵ) space and so analogues of 4.2(ii), (iii), (iv) and (v) hold. The crucial difference is in (i), so no simple analogue of the “quadratic” decomposition $Q(\mu)$ carries over to planar diffeomorphisms.

The remainder of the paper is devoted to the proof of Theorem 4.2.

5. Proof of Theorem 4.2. The proof of part (i) follows essentially from the geometry of f_μ^q when the saddle-node bifurcation at $\mu = s(p/q)$ occurs. The characterization of $Q(\mu)$ follows from the “quadratic-like” behavior of the maps $G_{\mu,i} = f_\mu^q|[t_i, t'_i]$ (see the discussion preceding the statement of the theorem) and the results of Jonker and Rand [1981a, b]. Also see van Strein [1980] or, for a different perspective, Nitecki [1982]. A nice summary of the decomposition theorem of Jonker and Rand (without proofs) is contained in Holmes and Whitley [1984].

The set \mathcal{S} is constructed by considering the behavior of g_μ on the complement of $W = \bigcup_{i=0}^{q-1} (t_i, t'_i) \cup G$ (Block [1973], Jakobson [1971]). Indeed, having removed the $q+1$ open intervals in W , $I - W$ consists of $q+2$ closed intervals $K_j, j = 0, \dots, q+1$, whose endpoints belong to the set $\{t_i, t'_i, 0, a, b, d\}$ We can establish explicitly how the intervals K_j are related combinatorially, thus establishing the structure of \mathcal{S} . Recall that the points t_i have been indexed so that $f_\mu(t_i) = t_{(i+1) \bmod q}$ for $i = 0, \dots, q - 1$. Recall further that t'_i was defined to be the smallest point greater than t_i such that $f_\mu^q(t'_i) = f_\mu^q(t_i)$. Using this information one easily shows that $f_\mu(t'_0) = t_1$ and $f_\mu(t'_i) = t'_{i+1}$ for $i \neq 0$.

If we define $\mathcal{S} = \bigcap_{i=0}^\infty f_\mu^{-i}(I - W)$, then we can associate to \mathcal{S} a transition matrix $A(\mathcal{S})$ defined by

$$A_{ij} = \begin{cases} 1 & \text{if } f_\mu(\mathbf{K}_{i-1}) \supset K_{j-1}, \\ 0 & \text{otherwise.} \end{cases}$$

(Be careful not to confuse indices! The K_i are labeled with $i = 0, \dots, q + 1$ but the indices of $A(\mathcal{S})$ are $i, j = 1, \dots, q + 2$.) Moreover, using the fact that the periodic

orbit $\{t_i\}_{i=0}^{q-1}$ is well ordered, it is not difficult to show that $A(\mathcal{S})$ must have the following form:

$$A(\mathcal{S}) = \begin{pmatrix} & K_0 & K_1 & \cdots & K_{p-2} & K_{p-1} & K_p & K_{p+1} & K_{p+2} & K_{p+3} & \cdots & K_{q+1} \\ K_0 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & 0 & 0 & \cdots & 0 \\ K_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ K_2 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots \\ K_{p-1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ K_p & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ K_{p+1} & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots \\ K_{q-1} & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ K_q & 0 & 0 & \cdots & 0 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ K_{q+1} & 1 & 1 & \cdots & 1 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Letting $\Sigma_A = \{a_0 a_1 \cdots | a_i \in \{0, \dots, q + 1\}, i \geq 0, A_{a_i a_{i+1}} = 1\}$ for each $x \in \mathcal{S}$ we can associate the sequence $h(x) \in \Sigma_A$ via $(h(x))_i = j$ if $f_\mu^i(x) \in K_j; i \geq 0, j = 0, \dots, q + 1$. The map $h: \mathcal{S} \rightarrow \Sigma_A$ is surjective by the definition of $A(\mathcal{S})$. Furthermore one can use the negative Schwarzian condition to show that for each $x \in \mathcal{S}$ there exists $n \geq 1$ such that $|(f_\mu^n)'(x)| > 1$. This in turn allows one to show that h is injective also. For details see Hockett [1986] (also Misiurewicz [1981b]). Finally if one provides Σ_A with the metric

$$d(\mathbf{a}, \mathbf{b}) = \sum_{i=0}^{\infty} |a_i - b_i|/3^i$$

then the map h is continuous with respect to this metric, hence h is a homeomorphism. (See, e.g., Guckenheimer and Holmes [1983].) This proves (i). We observe that the rotation numbers of orbits of Σ_A can be derived easily if the closed intervals K_j are divided into two classes—those in $[0, d](p + 1)$ and those in $[a, b](q - p + 1)$.

To prove (ii), consider the itinerary of $f_\mu(c), \nu_3 = \varepsilon(f_\mu(c))$, when $\mu = h(p/q)$. In this case $f_{h(p/q)}^q(c) = t'_0$, so from the proof of (i) we find

$$(5.1) \quad \nu_3(h(p/q)) = \varepsilon_0(t_1) \cdots \varepsilon_{q-2}(t_1) 2 (\varepsilon_0(t_1) \cdots \varepsilon_{q-2}(t_1) 1)'$$

where the ' denotes periodic repetition.

Let us denote $\nu_3(h(p/q))$ by ν_h . Now $t_0 \in I_1$, so the only '2' appearing in ν_h is the one identified in (5.1). Hence the periodically repeating part of ν_h consists only of 0's and 1's.

Now in ν_h choose any $\nu_h^i = 1, i > q - 1$, and observe that the itinerary obtained by replacing $\nu_h^i, \nu_h^{i+1}, \dots$ by G is admissible. Furthermore, since there is only a single '2' in ν_h , we have

$$\nu_h < \varepsilon_0(t_1) \cdots \varepsilon_{q-1}(t_1) 2 \cdots \nu_h^{i-1}(G)' < \varepsilon(f_{g(p/q)}(c)).$$

Since $f_\mu(c)$ varies continuously with μ , there exists $\mu \in (h(p/q), g(p/q))$ for which

$$(5.2) \quad \varepsilon(f_\mu(c)) = \nu_2(\mu) = \varepsilon_0(t_1) \cdots \varepsilon_{q-2}(t_1) 2 \cdots \nu_h^{i-1}(G)'.$$

Next note that $\epsilon(d) < (G)' < \epsilon(a)$ and that no admissible sequence lies between them. Thus we have

$$(5.3) \quad \begin{aligned} \nu_h < \epsilon_0(t_1) \cdots \epsilon_{q-2}(t_1) 2 \cdots \nu_h^{i-1} \epsilon(a) < \epsilon_0(t_1) \cdots \epsilon_{q-2}(t_1) 2 \cdots \nu_h^{i-1} (G)' \\ < \epsilon(t_1) \cdots \epsilon_{q-2}(t_1) 2 \cdots \nu_h^{i-1} \epsilon(d) < \epsilon(f_{g(p/q)}(c)), \end{aligned}$$

and so there exist values $\mu = g_i(p/q)$ and $\mu = g'_i(p/q)$ such that $f_{g_i(p/q)}^i(c) = a$ and $f_{g'_i(p/q)}^i(c) = d$. Moreover, $g_i(p/q) < g'_i(p/q)$. This proves the first part of (ii).

To prove the second part simply observe that there is a sequence of positive integers $i_1 < i_2 < i_3 < \cdots$ and a sequence of values $g_{i_1}(p/q) > g_{i_2}(p/q) > g_{i_3}(p/q) > \cdots$ such that $\mu = g_{i_k}(p/q)$ implies $f_{\mu}^{i_k}(c) = a$. Moreover as $k \rightarrow \infty$, $g_{i_k}(p/q) \rightarrow h(p/q)$. A similar statements holds for the $g'_{i_k}(p/q)$.

REMARK. The above tacitly assumes $q > 2$. The cases $q = 1$ and $q = 2$ are easily handled separately.

The proof of (v) uses arguments very similar to those used in (i) in establishing the structure of \mathcal{S} , so we omit it. Note, however, that in general one cannot expect to characterize the transition matrix of the subshift as we did $A(\mathcal{S})$.

It remains to prove (iii) and (iv). We shall need some additional terminology and a computational result.

DEFINITION 5.1. Let \mathbf{a} be a sequence of the symbols $\{0, G, 1, C, 2\}$ with $a_0 = C$ which for all $i \geq 1$ satisfies

$$\begin{aligned} a_i = 0 &\Rightarrow \nu_2 \leq \sigma^{i+1}(\mathbf{a}) \leq \nu_1, \\ a_i = G &\Rightarrow a_j = G \quad \text{for } j = i + 1, i + 2, \dots, \\ a_i \in \{1, C, 2\} &\Rightarrow \nu_2 \leq \sigma^{i+1}(\mathbf{a}) \leq \sigma(\mathbf{a}). \end{aligned}$$

Then we call \mathbf{a} a *circle kneading sequence*.

Note that $C\nu_3(\mu) = C\epsilon(f_{\mu}(c)) = \epsilon(c)$ is a circle kneading sequence. Our approach to proving (iii) and (iv) will be to construct circle kneading sequences \mathbf{a} with specified rotation numbers $R(\mathbf{a}) = p'/q'$. For this we will need

PROPOSITION 5.2. *Suppose that $Ca_1 \cdots a_n(0)'$ is a circle kneading sequence such that $a_n = 2$ (resp. $a_n = 1$) if the 2-parity of $a_1 \cdots a_{n-1}$ is odd and $a_n = 1$ (resp. $a_n = 2$) if the 2-parity of $a_1 \cdots a_{n-1}$ is even. Let $C(b_1 \cdots b_q)'$ be a periodic circle kneading sequence with $b_q \in \{1, 2\}$ such that $(b_1 \cdots b_q)' < a_1 \cdots a_n(0)'$.*

Then (i) there exists $M \geq 0$ such that $Ca_1 \cdots a_n(0)^m(b_1 \cdots b_q)'$ is a circle kneading sequence for all $m \geq M$;

(ii) there exists M such that for $m \geq M$ and for all $l \geq 1$,

$$C[a_1 \cdots a_n(0)^m(b_1 \cdots b_q)^{l-1}b_1 \cdots b_{q-1}C]'$$

is a circle kneading sequence;

(iii) if $C\mathbf{b}$ is any circle kneading sequence such that $\mathbf{b} < a_1 \cdots a_n(0)'$ then there exists $M \geq 0$ such that $Ca_1 \cdots a_n(0)^m\mathbf{b}$ is a circle kneading sequence for all $m \geq M$.

In each case the circle kneading sequence produced is larger (resp. smaller) than $a_1 \cdots a_n(0)'$.

PROOF. Pick $\bar{\mu} \in [\mu_0, \mu_1]$ such that $\nu_3(\bar{\mu}) = a_1 \cdots a_n(0)'$. Then $f_{\bar{\mu}}^{n+1}(c) = 0$. We can choose $\mu = \bar{\mu} + \epsilon$ such that $f_{\mu}^{n+1}(c) = 0 + \delta(\epsilon)$ where $\delta(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. Let x be the point in S^1 with itinerary $\epsilon(x) = (b_1 \cdots b_q)'$. This point exists since

$(b_1 \cdots b_q)' < a_1 \cdots a_n(0)'$ (Proposition 3.3). Now there exists a sequence $\{x_{-m}\}$ of preimages of x , all of which are contained in I_0 , accumulating on 0. The itinerary of x_{-m} is $\varepsilon(x_{-m}) = (0)^m(b_1 \cdots b_q)'$. Thus there exists $M \geq 0$ and a sequence $\varepsilon_m \rightarrow 0$ such that $\delta(\varepsilon_m) = x_{-m}$ for all $m \geq M$. We thus get a sequence of parameter values $\mu_m = \bar{\mu} + \varepsilon_m$ with the property $f_{\mu_m}^{n+1+m}(c) = x$. Thus $\nu_3(\mu_m) = a_1 \cdots a_n(0)^m(b_1 \cdots b_q)'$. This proves (i).

Next observe that the sequence $\mathbf{b}' = (b_i \cdots b_{q-1} \hat{b}_q)'$ is admissible for $f_{\bar{\mu}}$, where $\hat{b}_q = 1$ if $b_q = 2$ and $\hat{b}_q = 2$ if $b_q = 1$. (Note that if q is even then \mathbf{b}' represents a periodic point of period q or $q/2$; e.g.: $(0001)'$ vs $(0002)'$ and $(0102)'$ vs $(0101)'$.) Hence the sequence $\mathbf{b}^l = (b_1 \cdots b_q)^{l-1} b_1 \cdots b_{q-1} \varepsilon(c)$, $l \geq 1$, is admissible for $f_{\bar{\mu}}$. Thus there is a sequence $\{c_{-ql}\}$ of preimages of c accumulating on x , where x is the periodic point with itinerary $\varepsilon(x) = (b_1 \cdots b_q)'$. Given such a pre-image c_{-ql} there exists a sequence $\{c_{-ql-m}\}$ of preimages of c_{-ql} , all of which lie in I_0 , accumulating on 0. The itinerary of c_{-ql-m} is $\varepsilon(c_{-ql-m}) = (0)^m(b_1 \cdots b_q)^{l-1} b_1 \cdots b_{q-1} \varepsilon(c)$.

Now there exists M and a sequence $\varepsilon_{ql+m} \rightarrow 0$ as $m \rightarrow \infty$ such that for $m \geq M$, $\delta(\varepsilon_{ql+m}) = c_{-ql-m}$. So there is a sequence of parameter values $\mu_{ql+m} = \bar{\mu} + \varepsilon_{ql+m}$ such that $f_{\mu_{ql+m}}^{n+1}(c) = c_{-ql-m}$. Thus

$$\nu_3(\mu_{ql+m}) = [a_1 \cdots a_n(0)^m(b_1 \cdots b_q)^{l-1} b_1 \cdots b_{q-1} C]'$$

This proves (ii).

Statement (iii) is proved exactly as in (i), and the last part of the proposition is obvious since $a_1 \cdots a_n$ has even (resp. odd) 2-parity. \square

The properties of the circle kneading sequence $Ca_1 \cdots a_n(0)'$ of Proposition 5.2 are modeled on the behavior of ν_3 when exiting (resp. entering) one of the gaps constructed in the proof of Theorem 4.2(ii), though the results apply when entering or exiting *any* gap. This proposition provides an important computational tool for studying bifurcations to stable periodic orbits when exiting or entering a gap.

It is possible to show that for any $p/q \in [0, 1]$, $(p, q) = 1$, there are precisely two circle kneading sequences which correspond to periodic orbits which are well ordered (see Hockett [1986] for details). One of these sequences is the minimax sequence of Gambaudo et al. [1984] see also Bernhardt [1982]). Let \mathbf{a}_m denote this minimax sequence, i.e.,

$$\mathbf{a}_m = \min_{\mathbf{b} \in \Sigma_{p,q}} \left(\max_k \sigma^k(\mathbf{b}) \right)$$

where

$$\Sigma_{p,q} = \{ \mathbf{b} = b_0 b_1 \cdots | b_i = 0 \text{ or } 1, \mathbf{b} \text{ is } q\text{-periodic under the shift } \sigma \text{ and contains exactly } p \text{ 1's in one period} \}.$$

Then $\nu_\varepsilon(s(p/q)) = \mathbf{a}_m = (a_0 \cdots a_{q-2} 1)'$ is one of the two well ordered circle kneading sequences, the other being $(a_0 \cdots a_{q-2} 2)'$. Thus all of the sequences we shall now construct using Proposition 5.2 correspond to badly ordered periodic orbits.

We will now prove parts (iii)(a) and (b) of Theorem 4.2. The proofs of (iii)(c) and (d) are similar and are omitted.

For $\mu = g'_{i_{k+1}}(p/q)$, $\nu_3(\mu)$ takes the form

$$\nu_3(\mu) = \varepsilon_0(t_1) \cdots \varepsilon_{q-2}(t_1) 2(\varepsilon_0(t_1) \cdots \varepsilon_{q-2}(t_1) 1)^e \varepsilon_0(t_1) \cdots \varepsilon_{q-1}(t_1) 02(0)'$$

(see the proof of part (ii) of Theorem 4.2 above). Here $\varepsilon_g(t_1) = 1$, $g \leq q - 1$ and $e \geq 0$. Note that the only 2's appearing in $\nu_3(\mu)$ are the two shown, so the unparenthesized statements in Proposition 5.2 apply. Hence if $C(b_1 \cdots b_t)'$ is a circle kneading sequence with $b_t \in \{1, 2\}$ and $(b_1 \cdots b_t)' < \nu_3(\mu)$ for $\mu = g'_{i_{k+1}}(p/q)$, then for $m \geq M$ for some M and all $l \geq 1$, the sequence

$$C[\varepsilon_0(t_1) \cdots \varepsilon_{q-2}(t_1)2(\varepsilon_0(t_1) \cdots \varepsilon_{q-2}(t_1)1)^e \varepsilon(t_1) \cdots \varepsilon_{g-1}(t_1)02(0)^m (b_1 \cdots b_t)^{l-1} b_1 \cdots b_{t-1} C']$$

is a circle kneading sequence for f_μ when $\mu = g'_{i_{k+1}}(p/q) + \varepsilon_{tl+m}$ (using the notation of the proof of Proposition 5.2).

Suppose that $R((b_1 \cdots b_t)') = r/s$ (see Proposition 3.4 for the definition of R). Then $t = ks$ and $\sum_{i=1}^t \text{rot}(b_i) = kr$ for some $k \geq 1$. Suppose further that the finite sequence $\varepsilon_0(t_1) \cdots \varepsilon_{g-1}(t_1)$ of length g contains a total of f 1's and 2's. Then for $\mu = g'_{i_{k+1}}(p/q) + \varepsilon_{tl+m}$,

$$R(\nu_3(\mu)) = \frac{(1+e)p + f + 1 + krl}{(1+e)q + g + 2 + m + ksl}.$$

Now let $p'/q' \in (0, p/q)$. We seek to construct a sequence $\nu_3(\mu)$ such that $R(\nu_3(\mu)) = p'/q'$. In the formula for $R(\nu_3(\mu))$ above we are free to vary k , r/s and l . Note that in general M varies as a function of r/s and l , but since we are not particularly interested in ν_3 having minimal period, we may simply choose

$$M = \inf\{\tilde{M} | (0)^m (\varepsilon_0(t_1) \cdots \varepsilon_{q-2}(t_1)1)^{l-1} \varepsilon_0(t_1) \cdots \varepsilon_{q-2}(t_1) \varepsilon(c) \text{ is admissible for } f_\mu, \mu = g'_{i_{k+1}}(p/q) \text{ when } m \geq \tilde{M}\}.$$

That is, if we restrict the tail sequence $b_1 \cdots b_t$ so that $(b_1 \cdots b_t)' \leq \nu_3(s(p/q))$, say, then M can be bounded from below uniformly.

Now for fixed $m \geq M$ we wish to choose $r/s < p/q$, k and l such that

$$\frac{(1+e)p + f + 1 + krl}{(1+e)q + g + 2 + m + ksl} = \frac{p'}{q'}.$$

Hence we must satisfy $(1+e)p + f + 1 + krl = k'p'$ and $(1+e)q + g + 2 + m + ksl = k'q'$, for some $k' \geq 1$. Note that we are free to make a particular choice of k' . Letting $kl = L$, we must solve

$$(5.4) \quad (1+e)p + f + 1 + Lr = k'p', \quad (1+e)q + g + 2 + m + Ls = k'q'.$$

Solving (5.4) for r/s we obtain

$$(5.6) \quad \frac{r}{s} = \frac{k'p' - \alpha}{k'q' - \beta} = \frac{p' - \alpha/k'}{q' - \beta/k'}$$

where $\alpha = (1+e)p + f + 1$ and $\beta = (1+e)q + g + 2 + m$. Two things need to be satisfied in (5.6). First we must have that $k'p' - \alpha$ and $k'q' - \beta$ are positive integers. Second, we must have $(k'p' - \alpha)/(k'q' - \beta) \leq p/q$. From (5.6) it is evident that both can be satisfied for all k' sufficiently large. In particular there are infinitely many k' for which (5.6) yields an acceptable r/s .

Now in general $(k'p' - \alpha)/(k'q' - \beta)$ is not in lowest terms and the g.c.d. of the numerator and denominator, for a given k' , shows us how to choose L . This completes the proof of (iii)(a).

To prove (iii)(b) simply observe that by Proposition 5.2(iii) we can specify the tail sequence \mathbf{b} to be any sequence ending with $(G)'$ provided $\mathbf{b} < \nu_3(s(p/q))$, say, and m is sufficiently large.

Finally we turn to the proof of (iv). The first statement follows by using Proposition 5.2 as above to symbolically construct the desired periodic orbits and gaps. For the second statement note that at $\mu = g'(p/q)$, $f_\mu^q(c) = 0$, so $f_\mu^{q-1}(c) = a$, hence $\nu_3(g'(p/q)) = \varepsilon_0(t_1) \cdots \varepsilon_{q-3}(t_1)1(0)'$. (We have tacitly assumed $q \geq 2$. The case $q = 1$ is irrelevant since we defined μ_1 to be the smallest μ for which $f_\mu(c) = b$.) Assuming for the moment the truth of the last statement in (iv) (which characterizes $e(p/q)$ dynamically) we thus see that at $\mu = e(p/q)$, $\nu_3(e(p/q)) = \varepsilon_0(t_1) \cdots \varepsilon_{q-3}(t_1)1\varepsilon(t_k)$, where t_k is the smallest point in the periodic orbit $\{t_i\}_{i=0}^{q-1}$ created at $\mu = s(p/q)$. Note that $\nu_3(e(p/q))$ contains no 2's.

Consider the finite sequence $\varepsilon_0(t_1) \cdots \varepsilon_{q-3}(t_1)1$. Since $\varepsilon_0(t_1) \cdots \varepsilon_{q-1}(t_1) = \varepsilon_0(t_1) \cdots \varepsilon_{q-2}(t_1)1$ contains p 1's, the sequence $\varepsilon_0(t_1) \cdots \varepsilon_{q-3}(t_1)1$ contains p or $p - 1$ 1's, and hence contributes $p/(q - 1)$ or $(p - 1)/(q - 1)$ to the rotation number. We can rule out the case $(p - 1)/(q - 1)$ as follows. If this case occurs then $\varepsilon_0(t_1) \cdots \varepsilon_{q-2}(t_1)1 = \varepsilon_0(t_1) \cdots \varepsilon_{q-3}(t_1)11$ so that $\varepsilon(t_1) = (\varepsilon_0(t_1) \cdots \varepsilon_{q-3}(t_1)11)'$. Thus $\varepsilon(t_0) = (1\varepsilon_0(t_1) \cdots \varepsilon_{q-3}(t_1)1)'$ and we recall that t_0 is the point in the periodic orbit $\{t_i\}_{i=0}^{q-1}$ closest to c . Now $t_0 = f_\mu(t_{q-1})$ and $\varepsilon(t_{q-1}) = (11\varepsilon_0(t_1) \cdots \varepsilon_{q-3}(t_1))'$. Since $t_{q-1} < t_0$, we have $f_\mu(t_{q-1}) > t_{q-1} \in I_1$. Thus $\mu \geq s(1)$ and we have already noted that this case is irrelevant. Hence $\varepsilon_0(t_1) \cdots \varepsilon_{q-3}(t_1)1$ contains p 1's.

It is proved in Bernhardt [1982] that

$$\varepsilon(t_k) = (\varepsilon_0(t_k) \cdots \varepsilon_{q-1}(t_k))' = (\varepsilon_{q-1}(t_0) \cdots \varepsilon_0(t_0))'$$

i.e., the itinerary of t_k is obtained by writing down the itinerary of t_0 backwards. In particular, $\varepsilon_{q-1}(t_k) = \varepsilon_0(t_0) = 1$. Now observe that for $\mu = e(p/q)$, the sequence

$$\mathbf{a} = \varepsilon_0(t_1) \cdots \varepsilon_{q-3}(t_1)1(\varepsilon_0(t_k) \cdots \varepsilon_{q-1}(t_k))^{l-1}\varepsilon_0(t_k) \cdots \varepsilon_{q-2}(t_k)0\varepsilon(c)$$

is admissible for f_μ for $l \geq 1$. Hence there is a sequence $\{c_{-l}\}$ of preimages of c accumulating on t_k from below. For l sufficiently large, $c_{-l} = t_k - \varepsilon_l$, $\varepsilon_l \rightarrow 0$ as $l \rightarrow \infty$. Hence there exists $\mu_l = e(p/q) - \delta_l$, $\delta_l \rightarrow 0$ as $l \rightarrow \infty$, such that

$$\nu_3(\mu_l) = [\varepsilon_0(t_1) \cdots \varepsilon_{q-3}(t_1)1(\varepsilon_0(t_k) \cdots \varepsilon_{q-1}(t_k))^{l-1}\varepsilon_0(t_k) \cdots \varepsilon_{q-2}(t_k)0C]'$$

Now

$$R(\nu_3(\mu)l) = \frac{p + (l - 1)p + p}{(q - 1) + (l - 1)q + (q + 1)} = \frac{(l + 1)p}{(l + 1)q} = \frac{p}{q}.$$

It remains to prove the last statement in (iv), namely,

$$e(p/q) = \inf\{\mu > g'(p/q) \mid f_\mu^q(c) = t_k\},$$

where t_k is the smallest point in the periodic orbit $\{t_i\}_{i=0}^{q-1}$ created at $\mu = s(p/q)$. For $\mu \in [\mu_0, \mu_1]$ there exists a point $\xi \in I_0$ such that $f_\mu(\xi) = f_\mu(c)$. Define the upper interpolated map (Boyland [1986]) F_μ by

$$F_\mu(x) = \begin{cases} f_\mu(c) & \text{for } x \in [0, \xi] \cup [c, 1], \\ f_\mu(x) & \text{elsewhere.} \end{cases}$$

See Figure 6. The map F_μ is nondecreasing in the sense that if \bar{F}_μ is any lift of F_μ to \mathbf{R} , then $x < y$ implies $\bar{F}_\mu(x) \leq \bar{F}_\mu(y)$. Boyland shows that such maps

have a number of properties similar to those of homeomorphisms of the circle. In particular, every point in S^1 has the same rotation number under F_μ . In addition, one can show that $\rho(F_\mu) = \rho^+(f_\mu)$, and if $\rho(F_\mu) = p/q$ then there is a q -periodic point, θ , $\rho(\theta; f_\mu) = p/q$, such that $\{F_\mu^i(\theta)\}_{i=0}^{q-1} \cap \{[0, \xi] \cup (c, 1]\} = \emptyset$. That is, the orbit of θ does not fall into the flat piece.

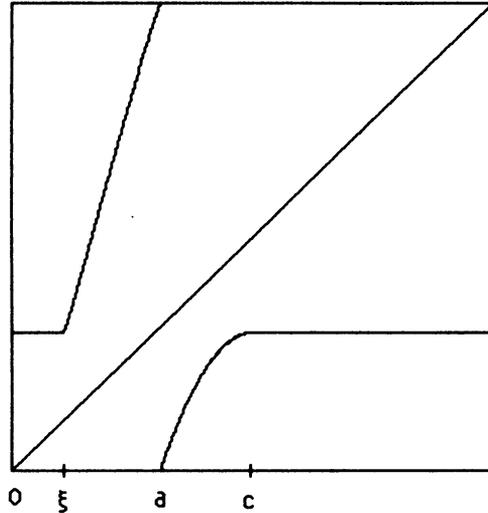


FIGURE 6. The upper interpolated map F_μ

Now by Theorem 4.2(i), $\rho(f_\mu; c) = p/q$ for $\mu \in [s(p/q), h(p/q)]$, so the p/q -step ends at some $\mu \geq h(p/q)$. Note also that for $\mu \in (h(p/q), g'(p/q))$, $\nu_3(\mu)$ either contains a 2 or ends with $(G)'$, so the orbit of the critical point falls into the flat piece in this case. Hence the p/q -step ends at some $\mu \geq g'(p/q)$.

Now as μ increases from $g'(p/q)$, $f_\mu^q(c)$ increases from 0. For μ sufficiently close to $g'(p/q)$, $f_\mu^q(c) \in [0, \xi]$, so again $\rho(c; f_\mu) \leq p/q$. We claim that when μ is such that $f_\mu^q(c) = t_k$, then $\xi = t_k$, so that for smaller μ , the orbit of the critical point falls into the flat piece but at that value of μ , $f_\mu^q(c)$ intersects the boundary of the flat piece. Denote this value by μ^* . Then $\rho(c; f_\mu) \leq p/q$ for $s(p/q) \leq \mu \leq \mu^*$ and hence $e(p/q) \geq \mu^*$.

We need to show that $\xi = t_k$ for $\mu = \mu^*$. Since $f_\mu(t_k) = t_{k+1}$, we need to show that $f_{\mu^*}(c) = t_{k+1}$. Now it takes $q - k$ iterates for t_k to return to t_0 , i.e., $f_\mu^{q-k}(t_k) = t_0$. Since $f_{\mu^*}^q(c) = f_{\mu^*}^{q-1}(f_{\mu^*}(c)) = t_k$, $f_{\mu^*}(c)$ lands on a point in the orbit of t_k which takes $q - 1$ iterates to return to t_k . That point is clearly t_{k+1} , i.e., $f_{\mu^*}(t_k) = f_{\mu^*}(c) = t_{k+1}$.

LEMMA 5.3. *There exists a sequence $s(p_l/q_l) \rightarrow \mu^*$ from above, with $p_l/q_l > p/q$, such that for $\mu = s(p_l/q_l)$ f_μ undergoes a saddle-node bifurcation to a q_l -periodic orbit of rotation number p_l/q_l .*

PROOF. Notice that $\nu_3(\mu^*) = \varepsilon_0(t_1) \cdots \varepsilon_{q-3}(t_1)1\varepsilon(t_k) = \varepsilon(t_{k+1})$. Since $p/q < 1$, $t_k \in I_0$, hence

$$\nu_3(\mu^*) = [\varepsilon_0(t_{k+1}) \cdots \varepsilon_{q-3}(t_{k+1})10]'$$

Now there is a sequence $\{c_{-l}\}$ of preimages of c accumulating on t_k from above with $\varepsilon(c_{-l}) = (\varepsilon_0(t_k) \cdots \varepsilon_{q-1}(t_k))^l \varepsilon_0(t_k) \cdots \varepsilon_{q-2}(t_k) \varepsilon(c)$. For l sufficiently large, $c_{-l} = t_k + \varepsilon_l$, $\varepsilon_l \rightarrow 0$ as $l \rightarrow \infty$. Hence there is a sequence of preimages $\{c'_{-l}\}$ of c accumulating on t_{k+1} with itinerary

$$\mathbf{a}_l = \varepsilon_0(t_{k+1}) \cdots \varepsilon_{q-3}(t_{k+1}) 1 (\varepsilon_0(t_k) \cdots \varepsilon_{q-1}(t_k))^l \varepsilon_0(t_k) \cdots \varepsilon_{q-2}(t_k) \varepsilon(c).$$

Again for l sufficiently large, $c'_{-l} = t_{k+1} + \varepsilon'_l$, $\varepsilon'_l \rightarrow 0$ and $l \rightarrow \infty$. so there exists $\mu_l = \mu^* + \delta'_l$ such that

$$\nu_3(\mu_l) = [\varepsilon_0(t_{k+1}) \cdots \varepsilon_{q-3}(t_{k+1}) 1 (\varepsilon_0(t_k) \cdots \varepsilon_{q-1}(t_k))^l \varepsilon_0(t_k) \cdots \varepsilon_{q-2}(t_k) C]^l.$$

Now

$$R(\nu_3(\mu_l)) = \frac{p + lp + p}{(q-1) + lq + q} = \frac{(l+2)p}{(l+2)q-1} > \frac{p}{q}. \quad \square$$

Hence $\mu^* = e(p/q)$ since $\rho^+(\mu) = p/q$ for $s(p/q) \leq \mu \leq \mu^*$ and $\rho^+(\mu) > p/q$ for $\mu > \mu^*$. This concludes the proof of Theorem 4.2.

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