ABSTRACT. Let $\tau$ denote the Thurston norm on $H_2(N; \mathbb{R})$, where $N$ is a closed, oriented, irreducible, atoroidal three-manifold $N$. U. Oertel defined a taut oriented branched surface to be a branched surface with the property that each surface it carries is incompressible and $\tau$-minimizing for the (nontrivial) homology class it represents. Given $\varphi$, a face of the $\tau$-unit sphere in $H_2(N; \mathbb{R})$, Oertel then asks: is there a taut oriented branched surface carrying surfaces representing every integral homology class projecting to $\varphi$? In this article, an example is constructed for which the answer is negative.

0. Introduction. Let $\tau$ denote the Thurston norm on $H_2(N; \mathbb{R})$ for a closed, oriented, irreducible, atoroidal three-manifold $N$. Following the definition of U. Oertel, let a taut oriented branched surface be a branched surface $B$ with the property that each surface carried by $B$ is incompressible and $\tau$-minimizing for the (nontrivial) homology class it represents. If $\varphi$ is a face of the $\tau$-unit sphere in $H_2(N; \mathbb{R})$, we say that $\varphi$ is spanned by a branched surface $B$ if $B$ carries surfaces representing every integral homology class which projects to $\varphi$. Let $V_1, \ldots, V_k$ be the vertices of $\varphi$, and let $B$ be a taut oriented branched surface spanning $\varphi$. Then $B$ carries surfaces $F_1, \ldots, F_k$ representing integral classes which project to $V_1, \ldots, V_k$. Further, if $+$ denotes oriented cut-and-paste, then $B$ must also carry $F_1 + \cdots + F_k$, which by definition is then incompressible and $\tau$-minimizing. In this note, we present an example of $N$ and $\varphi$ with the property that any $\tau$-minimizing surfaces representing classes projecting to certain vertices of $\varphi$ have a compressible cut-and-paste sum. Thus, no branched surface spanning $\varphi$ could be a taut oriented branched surface.

The construction of $N$ is in two parts. In §1, we construct a manifold with boundary $M$ which in fact would serve as $N$ if it were closed. In §2, we rectify this by gluing two copies of $M$ together to get $N$ without changing the relevant properties of $M$.

1. Construction of $M$. Let $f_i$ ($i = 1, 2, 3$) be copies of a twice punctured torus. Attach $f_i \times I$ ($i = 1, 2, 3$) to a solid torus $V$, gluing $(\partial f_i) \times I$ along longitudinal annuli of $\partial V$ according to the schematic in Figure 1 ($I = [0, 1]$). Call the result $M$. Let $F_i$ be the genus-2 surface formed from $f_i \times \frac{1}{2}$ by attaching an essential annulus in $V$ to the boundary components of $f_i \times \frac{1}{2}$ (see $F_i$ in Figure 1). $M$ has two boundary components, $\partial_0 M$ and $\partial_1 M$, where $(f_i \times j) \subset \partial_j M$. Orient $M, F_i$ so
that the transverse direction to \( f_i \times \frac{1}{2} \) runs from \( f_i \times 0 \) to \( f_i \times 1 \). Consider \( \partial V \) as the union of two sets of six longitudinal annuli apiece, \( \omega = \bigcup (\partial f_i) \times I \) and \( \rho = \partial V \setminus \omega \).

**Lemma 1.** (i) \( M \) is irreducible and atoroidal.

(ii) \( \partial_0 M \) and \( \partial_1 M \) are \( x \)-minimizing in \( H_2(M) \).

(iii) If \( H_i \) is a genus-2 surface in \( M \) with \([H_i] = [F_i] \in H_2(M)\), then \( H_i \) is isotopic to \( F_i \).

(iv) If \( H_i \) (\( i = 1, 2, 3 \)) is as above, \( H_1 + H_2 + H_3 \) contains a closed component entirely in \( V \), and is therefore compressible.

**Proof.** (i) is obvious.

(ii) Any \( \chi_- \)-minimizing surface \( G \) homologous to (say) \( \partial_0 M \) must intersect \( V \) in annuli and \( f_i \times I \) in a surface with Euler characteristic \(-2\). Therefore, \( \chi_-(G) \geq 6 = \chi_- (\partial_0 M) \).

(iii) \( H_i \cap (f_j \times I) \) (\( i \neq j \)) must consist of annuli parallel to \((\partial f_i) \times I\). These can be removed by isotopy. Similarly, \( H_i \cap ((\partial f_i) \times I) \) may be made a single simple closed curve.

(iv) The isotopy in (iii) may be performed on \( \omega \) rather than on \( H_i \). Thus, assume that \( C_i = V \cap H_i \) is an annulus and that \((\bigcap H_i) = (\bigcap C_i) \subset V\). If \( C_1 + C_2 \) contains a closed component \( W \), then either \( C_3 \cap W = \emptyset \) or \( C_3 \) separates \( W \). In the latter case, one “half” or the other gives rise to a closed component of \( C_1 + C_2 + C_3 \).

Assume, therefore, that \( C_1 + C_2 \) is two annuli \( D_+ \) and \( D_- \), with \( D_+ \) (resp., \( D_- \)) cutting out a solid torus \( Y_+ \) (resp., \( Y_- \)) from \( V \) with the transverse direction pointing out of \( Y_+ \) (resp., into \( Y_- \)). Now \( C_1 \cup C_2 \) separates the two components of \( \partial C_3 \), so \( C_3 \cap (C_1 + C_2) \) is not empty: say that \( C_3 \cap D_+ \neq \emptyset \). Let the subtorus of
The transverse direction of $C_3$ pointing out be called $X$. Then $Y_+ \cap X \neq \emptyset$ and $Y_+ \cap X \cap \partial D_+ = \emptyset$, so $D_+ + C_3$ has a closed component. Q.E.D.

The following lemma will be needed in the next section.

**Lemma 2.** Let $\gamma_0$ be a simple closed curve in $\partial_0 M$ such that $[\gamma_0] \neq 1 \in \pi_1(\partial_0 M)$ and one of the following holds:

(i) $\exists$ a simple closed curve $\gamma_1 \subset \partial_1 M$ such that $\gamma_0 \cup \gamma_1 = \partial C$ for an incompressible, $\partial$-incompressible annulus $C \subset M$;

(ii) $\gamma_0$ is isotopic in $M$ to a curve $\gamma_1 \subset \partial_1 M$; or

(iii) $\gamma_0$ is isotopic in $M$ to a curve $\delta \subset F_i$ for some $i$.

Then $\exists$ an isotopy of $\gamma_0$ in $\partial_0 M$ such that, after the isotopy, $\gamma_0 \cap V = \emptyset$.

**Proof.**

(i) Let $D$ be an annulus in the collection $\omega$, and consider $C \cap D$. Use $\partial$-incompressibility of $C$ to remove inessential arcs of $C \cap D$. Suppose $a$ is an essential arc of $C \cap D$; then there must be other, parallel arcs of $C \cap \omega$ in $C$. Choose the component $E$ of $C \cap \omega$ which contains $a$ in its boundary and which lies in $V$. Let $\beta$ be the other arc of $C \cap \omega$ in $\partial E$. If $a \subset (\partial f_1) \times I$, the fact that spanning arcs of $(\partial f_1) \times I (k \neq i)$ represent different classes of $H_1(M, \partial M)$ from $a$ implies that $\beta \subset (\partial f_1) \times I$ as well. Thus $E$ may be pushed out of $V$ into $f_k \times I$. Finish the proof by induction.

(ii) Use the Generalized Dehn’s Lemma [SW] to find an embedded annulus as in the statement of condition (i).

(iii) The isotopy can be extended from $F_i$ to $\partial_1 M$ by noticing there is a “reflection” of $M$ in $F_i$. That is, if $i = 1$, the map taking $f_1 \times j \rightarrow f_1 \times (1 - j)$, $f_2 \times j \rightarrow f_3 \times (1 - j)$, and $f_3 \times j \rightarrow f_2 \times (1 - j)$ can be extended to $V$ as a homeomorphism. Q.E.D.

2. **Construction of $N$.** A component $\partial_2 M$ of $\partial M$ can be seen in Figure 2. Let $\eta_j : \partial_2 M \rightarrow \partial_2 M$ be the product of Dehn twists in the two bold curves shown in Figure 2. Notice that for any simple closed curve $\gamma$ in $\partial_2 M$ with $\gamma \cap \partial \rho = \emptyset$ and $[\gamma] \neq 1 \in \pi_1(\partial_2 M)$, $\eta_j(\gamma) \cap \partial \rho \neq \emptyset$ and each arc of $\eta_j(\gamma) \cap \partial \rho$ is essential in the component of $\partial_2 M \setminus \partial \rho$ which contains it. (In general, any curves with these properties suffice for this construction.) Define $N$ to be two copies of $M$ glued along their boundaries by $\eta_j$. Pick one copy of $M \subset N$ to be referred to as $M$; the other will be $M'$.

**Lemma 3.** $N$ is irreducible and atoroidal. $\partial_2 M$ is $x$-minimizing for $[\partial_2 M] \in H_2(N)$.

**Proof.** Since the $\partial_2 M$ are incompressible and $M$ is irreducible, $N$ is irreducible. Consider a torus $T \subset N$ in general position with respect to $\partial M$. Remove inessential curves of $T \cap \partial M$; then $T \cap \partial M$ consists of parallel essential curves on $T$. Now note that, if $D$ is an annular component of $T \setminus \partial M$ lying in $M$, either $D$ is $(\partial-)compressible or both adjacent annuli of $T \setminus \partial M$ are $(\partial-)compressible in $M'$ (this uses Lemma 2 and the construction of $N$). Thus, either $T$ is compressible or $T \cap \partial M$ may be isotopically reduced until $T$ lies in $M$ or $M'$ and bounds a solid torus.

If $H_j$ is incompressible and $x$-minimizing for $[\partial_j M] \subset N$, an argument of Gabai (Lemma 3.6 of [G]) shows that $H_j$ may be taken as disjoint from $\partial M$. Then $H_j \subset M$ or $M'$, implying that $x(H_j) = x(\partial_j M)$. Q.E.D.
Lemma 4. Let $H_i$ be a genus-2 surface in $N$ with $[H_i] = [F_i] \in H_2(N)$. Then there exists an isotopy of $H_i$ in $N$ taking $H_i$ to $F_i$.

**Proof.** For convenience, construct the cover $\tilde{\Pi}: \tilde{N} \to N$ using countably many copies of $M$ glued alternately using $\eta_0$ and $\eta_1$, indexing the copies of $M$ so that $\Pi(M_0) = M_1, \Pi(M_{\pm 1}) = M', \Pi(M_{\pm 2}) = M$, etc. Let $\tilde{F}_i$ be the lift of $F_i$ into $M_0$ and $\tilde{H}_i$ the corresponding lift of $H_i$. Let $k_+$ be the largest index $n$ such that $M_n \cap \tilde{H}_i \neq \emptyset$; define $k_-$ analogously. Induct on $k = k_+ - k_-$. $k = 0$ is Lemma 1(iii).

Without loss of generality, assume that $n = k_+ > 0$. Let $Y = \tilde{H}_i \cap M_n$. Use the product structure on $(f_j \times I) \subset M_n$ to push $Y \cap ((f_j \times I) \subset M_n)$ into $M_{n-1}$. Similarly, push any $\partial$-parallel components of $Y$ remaining in $V \subset M_n$ into $M_{n-1}$. Let what is left of $Y$ in $M_n$ still be called $Y$.

Claim. $Y = \emptyset$.

If not, then $[Y] = 0 \in H_2(M_n, \partial M_n)$ but each component of $Y$ is nontrivial in homology. This forces $Y$ to be pairs of essential annuli in $V \subset M_n$ with boundary components lying exclusively on $\partial_0 M_n$ or $\partial_1 M_n$: say $\partial_0 M_n$. Let $\tilde{H}_i'$ be $\tilde{H}_i$ after surgery on $\tilde{H}_i$ along $\partial_0 M_n$ and after discarding the resulting toral components of $M_n$ (see Figure 3).

$\tilde{H}_i'$ has a smaller $k$ than $\tilde{H}_i$, so by induction $\tilde{H}_i'$ is isotopic to $\tilde{F}_i$; in particular, the curves of $\partial Y$ are isotopic either to curves in $\partial_1 M_{n-1}$ (if $n > 1$) or in $\tilde{F}_i$ (if $n = 1$). By Lemma 2 and the map $\eta_0$, $\partial Y$ would have to have nontrivial intersection with $\partial \rho \subset M_n$. This contradiction establishes the Claim and finishes the proof. Q.E.D.

Theorem. There is a face of the $x$-unit sphere in $H_2(N)$ which is not spanned by a taut oriented branched surface.
PROOF. Since \( F_1, F_2, F_3 \) and \( F_1 + F_2 + F_3 \) are all \( x \)-minimizing representatives of their respective classes in \( H_2(N) \), their classes project to the same face \( \varphi \) of the unit sphere. Let \( \mathcal{B} \) be a branched surface spanning \( \varphi \), and let \( H_i \subset N \) be surfaces carried by \( \mathcal{B} \) such that \( [H_i] = [F_i] \in H_2(N) \) and each \( H_i \) has genus 2. Then the proof of Lemma 4 constructs an isotopy of \( \partial_0 M \) and \( \partial_1 M \) in \( N \) which results in \( H_i \subset M \). By Lemma 1, \( H_1 + H_2 + H_3 \) contains a homologically trivial component \( D \) in \( V \). \( \mathcal{B} \) carries \( \sum H_i \), so it must carry \( D \). Q.E.D.

REFERENCES


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