PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 102, Number 4, April 1988

THURSTON NORM AND TAUT BRANCHED SURFACES

B. D. STERBA-BOATWRIGHT

(Communicated by Haynes R. Miller)

ABSTRACT. Let x denote the Thurston norm on $H_2(N;\mathbf{R})$, where N is a closed, oriented, irreducible, atoroidal three-manifold N. U. Oertel defined a taut oriented branched surface to be a branched surface with the property that each surface it carries is incompressible and x-minimizing for the (nontrivial) homology class it represents. Given φ, a face of the x-unit sphere in $H_2(N;\mathbf{R})$, Oertel then asks: is there a taut oriented branched surface carrying surfaces representing every integral homology class projecting to φ? In this article, an example is constructed for which the answer is negative.

0. Introduction. Let x denote the Thurston norm on $H_2(N;\mathbf{R})$ for a closed, oriented, irreducible, atoroidal three-manifold N. Following the definition of U. Oertel, let a taut oriented branched surface be a branched surface $\mathbf{B}$ with the property that each surface carried by $\mathbf{B}$ is incompressible and x-minimizing for the (nontrivial) homology class it represents. If φ is a face of the x-unit sphere in $H_2(N;\mathbf{R})$, we say that φ is spanned by a branched surface $\mathbf{B}$ if $\mathbf{B}$ carries surfaces representing every integral homology class which projects to φ. Let $V_1,\dots,V_k$ be the vertices of φ, and let $\mathbf{B}$ be a taut oriented branched surface spanning φ. Then $\mathbf{B}$ carries surfaces $F_1,\dots,F_k$ representing integral classes which project to $V_1,\dots,V_k$. Further, if $+$ denotes oriented cut-and-paste, then $\mathbf{B}$ must also carry $F_1+\cdots+F_k$, which by definition is then incompressible and x-minimizing. In this note, we present an example of N and φ with the property that any x-minimizing surfaces representing classes projecting to certain vertices of φ have a compressible cut-and-paste sum. Thus, no branched surface spanning φ could be a taut oriented branched surface.

The construction of N is in two parts. In §1, we construct a manifold with boundary M which in fact would serve as N if it were closed. In §2, we rectify this by gluing two copies of M together to get N without changing the relevant properties of M.

1. Construction of M. Let f_i $(i=1,2,3)$ be copies of a twice punctured torus. Attach $f_i\times I$ $(i=1,2,3)$ to a solid torus V, gluing $(\partial f_i)\times I$ along longitudinal annuli of ∂V according to the schematic in Figure 1 $(I=[0,1])$. Call the result M. Let F_i be the genus-2 surface formed from $f_i\times\frac{1}{2}$ by attaching an essential annulus in V to the boundary components of $f_i\times\frac{1}{2}$ (see F_1 in Figure 1). M has two boundary components, ∂_0M and ∂_1M, where $(f_i\times j)\subset\partial_jM$. Orient M, F_i so

Received by the editors December 15, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 57N10; Secondary 57M99.
Key words and phrases. Branched surface, taut oriented branched surface, Thurston norm, x-minimizing surface.

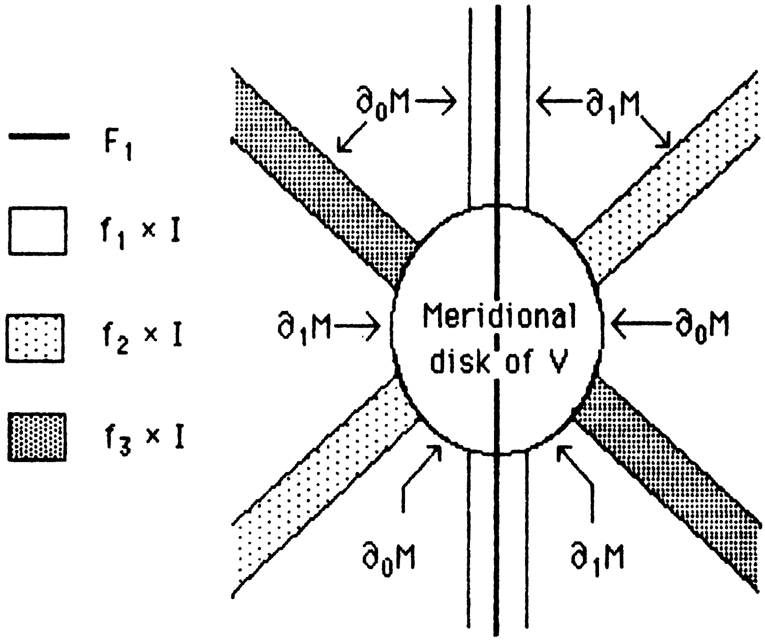

FIGURE 1

that the transverse direction to $f_i \times \frac{1}{2}$ runs from $f_i \times 0$ to $f_i \times 1$. Consider ∂V as the union of two sets of six longitudinal annuli apiece, $\omega = \bigcup(\partial f_i) \times I$ and $\rho = \overline{\partial V \backslash \omega}$.

LEMMA 1. (i) *M is irreducible and atoroidal.*

(ii) *$\partial_0 M$ and $\partial_1 M$ are x-minimizing in $H_2(M)$.*

(iii) *If H_i is a genus-2 surface in M with $[H_i] = [F_i] \in H_2(M)$, then H_i is isotopic to F_i.*

(iv) *If H_i $(i = 1, 2, 3)$ is as above, $H_1 + H_2 + H_3$ contains a closed component entirely in V, and is therefore compressible.*

PROOF. (i) is obvious.

(ii) Any χ_--minimizing surface G homologous to (say) $\partial_0 M$ must intersect V in annuli and $f_i \times I$ in a surface with Euler characteristic -2. Therefore, $\chi_-(G) \geq 6 = \chi_-(\partial_0 M)$.

(iii) $H_i \cap (f_j \times I)$ $(i \neq j)$ must consist of annuli parallel to $(\partial f_j) \times I$. These can be removed by isotopy. Similarly, $H_i \cap ((\partial f_i) \times I)$ may be made a single simple closed curve.

(iv) The isotopy in (iii) may be performed on ω rather than on H_i. Thus, assume that $C_i = V \cap H_i$ is an annulus and that $(\bigcap H_i) = (\bigcap C_i) \subset V$. If $C_1 + C_2$ contains a closed component W, then either $C_3 \cap W = \varnothing$ or C_3 separates W. In the latter case, one "half" or the other gives rise to a closed component of $C_1 + C_2 + C_3$.

Assume, therefore, that $C_1 + C_2$ is two annuli D_+ and D_-, with D_+ (resp., D_-) cutting out a solid torus Y_+ (resp., Y_-) from V with the transverse direction pointing out of Y_+ (resp., into Y_-). Now $C_1 \cup C_2$ separates the two components of ∂C_3, so $C_3 \cap (C_1 + C_2)$ is not empty: say that $C_3 \cap D_+ \neq \varnothing$. Let the subtorus of

$V \backslash C_3$ with the transverse direction of C_3 pointing out be called X. Then $Y_+ \cap X \neq \emptyset$ and $Y_+ \cap X \cap \partial D_+ = \emptyset$, so $D_+ + C_3$ has a closed component. Q.E.D.

The following lemma will be needed in the next section.

LEMMA 2. *Let γ_0 be a simple closed curve in $\partial_0 M$ such that $[\gamma_0] \neq 1 \in \pi_1(\partial_0 M)$ and one of the following holds:*

(i) *$\exists$ a simple closed curve $\gamma_1 \subset \partial_1 M$ such that $\gamma_0 \cup \gamma_1 = \partial C$ for an incompressible, ∂-incompressible annulus $C \subset M$;*

(ii) *γ_0 is isotopic in M to a curve $\gamma_1 \subset \partial_1 M$; or*

(iii) *γ_0 is isotopic in M to a curve $\delta \subset F_i$ for some i.*

Then $\exists$ an isotopy of γ_0 in $\partial_0 M$ such that, after the isotopy, $\gamma_0 \cap \partial V = \emptyset$.

PROOF. (i) Let D be an annulus in the collection ω, and consider $C \cap D$. Use ∂-incompressibility of C to remove inessential arcs of $C \cap D$. Suppose α is an essential arc of $C \cap D$; then there must be other, parallel arcs of $C \cap \omega$ in C. Choose the component E of $C \backslash \omega$ which contains α in its boundary and which lies in V. Let β be the other arc of $C \cap \omega$ in ∂E. If $\alpha \subset (\partial f_i) \times I$, the fact that spanning arcs of $(\partial f_k) \times I$ $(k \neq i)$ represent different classes of $H_1(M, \partial M)$ from α implies that $\beta \subset (\partial f_i) \times I$ as well. Thus E may be pushed out of V into $f_k \times I$. Finish the proof by induction.

(ii) Use the Generalized Dehn's Lemma [**SW**] to find an embedded annulus as in the statement of condition (i).

(iii) The isotopy can be extended from F_i to $\partial_1 M$ by noticing there is a "reflection" of M in F_i. That is, if $i = 1$, the map taking $f_1 \times j \to f_1 \times (1-j)$, $f_2 \times j \to f_3 \times (1-j)$, and $f_3 \times j \to f_2 \times (1-j)$ can be extended to V as a homeomorphism. Q.E.D.

2. Construction of N. A component $\partial_j M$ of ∂M can be seen in Figure 2. Let $\eta_j \colon \partial_j M \to \partial_j M$ be the product of Dehn twists in the two bold curves shown in Figure 2. Notice that for any simple closed curve γ in $\partial_j M$ with $\gamma \cap \partial \rho = \emptyset$ and $[\gamma] \neq 1 \in \pi_1(\partial_j M)$, $\eta_j(\gamma) \cap \partial \rho \neq \emptyset$ and each arc of $\eta_j(\gamma) \backslash \partial \rho$ is essential in the component of $\partial_j M \backslash \partial \rho$ which contains it. (In general, any curves with these properties suffice for this construction.) Define N to be two copies of M glued along their boundaries by η_j. Pick one copy of $M \subset N$ to be referred to as M; the other will be M'.

LEMMA 3. *N is irreducible and atoroidal. $\partial_j M$ is x-minimizing for $[\partial_j M] \in H_2(N)$.*

PROOF. Since the $\partial_j M$ are incompressible and M is irreducible, N is irreducible. Consider a torus $T \subset N$ in general position with respect to ∂M. Remove inessential curves of $T \cap \partial M$; then $T \cap \partial M$ consists of parallel essential curves on T. Now note that, if D is an annular component of $T \backslash \partial M$ lying in M, either D is $(\partial-)$compressible or both adjacent annuli of $T \backslash \partial M$ are $(\partial-)$compressible in M' (this uses Lemma 2 and the construction of N). Thus, either T is compressible or $T \cap \partial M$ may be isotopically reduced until T lies in M or M' and bounds a solid torus.

If H_j is incompressible and x-minimizing for $[\partial_j M]$ in N, an argument of Gabai (Lemma 3.6 of [**G**]) shows that H_j may be taken as disjoint from ∂M. Then $H_j \subset M$ or M', implying that $x(H_j) = x(\partial_j M)$. Q.E.D.

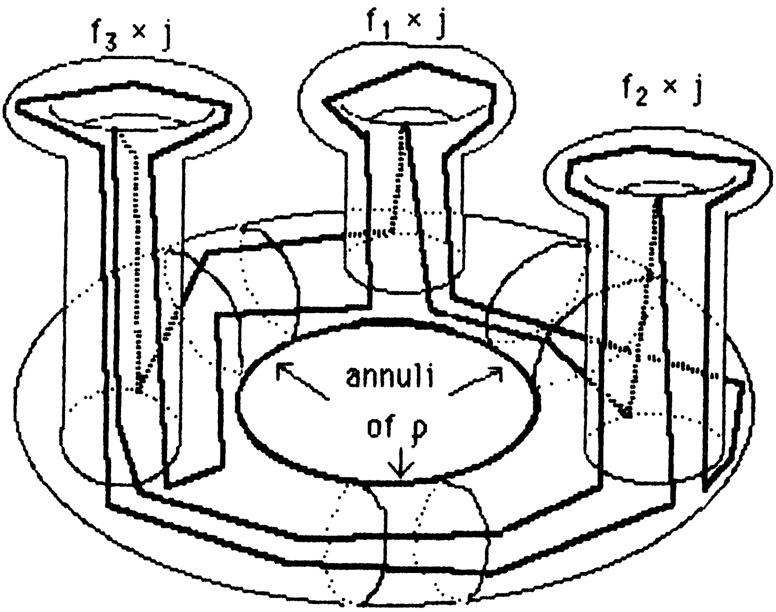

FIGURE 2

LEMMA 4. *Let H_i be a genus-2 surface in N with $[H_i] = [F_i] \in H_2(N)$. Then $\exists$ an isotopy of H_i in N taking H_i to F_i.*

PROOF. For convenience, construct the cover $\Pi\colon \tilde{N} \to N$ using countably many copies of M glued alternately using η_0 and η_1, indexing the copies of M so that $\Pi(M_0) = M$, $\Pi(M_{\pm 1}) = M'$, $\Pi(M_{\pm 2}) = M$, etc. Let $\tilde{F}_i$ be the lift of F_i into M_0 and $\tilde{H}_i$ the corresponding lift of H_i. Let k_+ be the largest index n such that $M_n \cap \tilde{H}_i \neq \varnothing$; define k_- analogously. Induct on $k = k_+ - k_-$. $k = 0$ is Lemma 1(iii).

Without loss of generality, assume that $n = k_+ > 0$. Let $Y = \tilde{H}_i \cap M_n$. Use the product structure on $(f_j \times I) \subset M_n$ to push $Y \cap ((f_j \times I) \subset M_n)$ into M_{n-1}. Similarly, push any ∂-parallel components of Y remaining in $V \subset M_n$ into M_{n-1}. Let what is left of Y in M_n still be called Y.

Claim. $Y = \varnothing$.

If not, then $[Y] = 0 \in H_2(M_n, \partial M_n)$ but each component of Y is nontrivial in homology. This forces Y to be pairs of essential annuli in $V \subset M_n$ with boundary components lying exclusively on $\partial_0 M_n$ or $\partial_1 M_n$: say $\partial_0 M_n$. Let $\tilde{H}_i'$ be $\tilde{H}_i$ after surgery on $\tilde{H}_i$ along $\partial_0 M_n$ and after discarding the resulting toral components of M_n (see Figure 3).

$\tilde{H}_i'$ has a smaller k than $\tilde{H}_i$, so by induction $\tilde{H}_i'$ is isotopic to $\tilde{F}_i$; in particular, the curves of ∂Y are isotopic either to curves in $\partial_1 M_{n-1}$ (if $n > 1$) or in $\tilde{F}_i$ (if $n = 1$). By Lemma 2 and the map η_0, ∂Y would have to have nontrivial intersection with $\partial\rho \subset M_n$. This contradiction establishes the Claim and finishes the proof. Q.E.D.

THEOREM. *There is a face of the x-unit sphere in $H_2(N)$ which is not spanned by a taut oriented branched surface.*

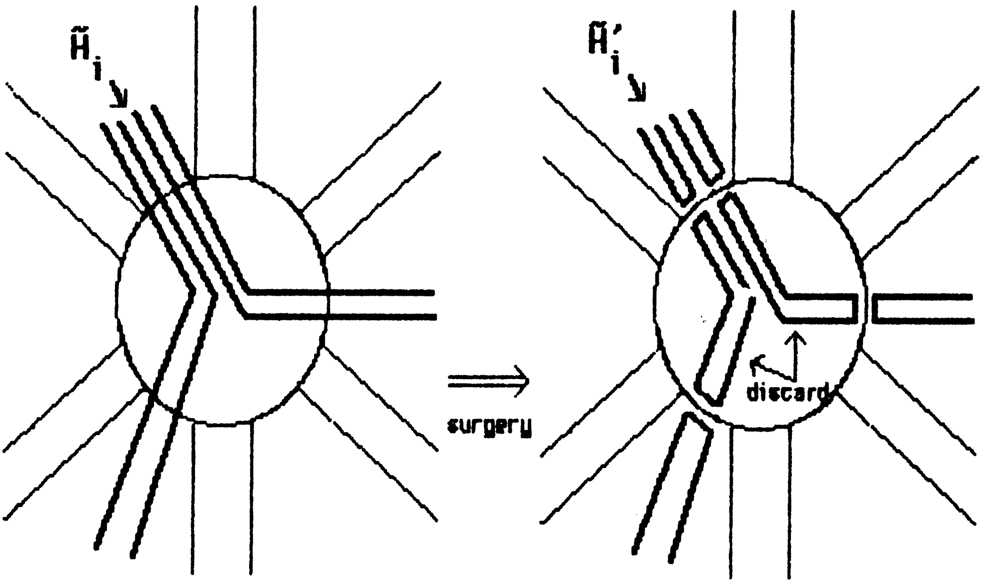

FIGURE 3

PROOF. Since F_1, F_2, F_3 and $F_1 + F_2 + F_3$ are all x-minimizing representatives of their respective classes in $H_2(N)$, their classes project to the same face φ of the unit sphere. Let $\mathbf{B}$ be a branched surface spanning φ, and let $H_i \subset N$ be surfaces carried by $\mathbf{B}$ such that $[H_i] = [F_i] \in H_2(N)$ and each H_i has genus 2. Then the proof of Lemma 4 constructs an isotopy of $\partial_0 M$ and $\partial_1 M$ in N which results in $H_i \subset M$. By Lemma 1, $H_1 + H_2 + H_3$ contains a homologically trivial component D in V. $\mathbf{B}$ carries $\sum H_i$, so it must carry D. Q.E.D.

REFERENCES

[G] D. Gabai, *Foliations and the topology of 3-manifolds*, J. Differential Geom. **18** (1983), 445–503.

[SW] A. Shapiro and H. Whitehead, *A proof and extension of Dehn's lemma*, Bull. Amer. Math. Soc. **64** (1958), 174–178.

DIVISION OF PHYSICAL AND BIOLOGICAL SCIENCES, ST. EDWARD'S UNIVERSITY, AUSTIN, TEXAS 78704