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THE PSEUDOCOMPACT EXTENSION αX

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ABSTRACT. For any Tychonoff space we define $\alpha X = (\beta X - vX) \cup X = \beta X - (vX - X)$. We show that αX is the smallest pseudocompactification Y of X contained is βX such that every free hyperreal z-ultrafilter on X converges in Y and is the largest pseudocompactification Y of X contained in βX such that every point in $Y - X$ is contained in a zero set of Y which does not intersect X. A space S is defined to be α-embedded in a space X if $\alpha S \subset \beta X$. Properties of α-embeddings and its relation to v-embeddings of Blair C^*-embeddings, C-embeddings, and well-embeddings are investigated.

For instance, if S is α-embedded and dense in X, S is fully well-embedded (for $P, R \subset X$, where $S \subset P \subset R \subset X$, P is well-embedded in R) in X iff $\alpha X - \alpha S = X - S$.

1. Introduction. For a Tychonoff space X that is not pseudocompact there are at least 2^{2^c} distinct pseudocompactifications of X contained in βX (see Example 1). Here we focus on a particular pseudocompactification $\alpha X = (\beta X - vX) \cup X$ and study its characterizations among pseudocompactifications and further elucidate the structure of βX. In particular, αX is the smallest pseudocompactification Y of X contained in βX such that every free hyperreal z-ultrafilter on X converges in Y. On the other hand, αX is the largest pseudocompactification Y of X, such that $Y \subset \beta X$ and every point of $Y - X$ is contained in a zero set of Y which does not intersect X. We then define an embedding of a space S in X, $S \subset X$ as an α-embedding if $\alpha S \subset \beta X$, and study its properties and its relation to well-embedding, C^*, C, and v-embedding of Blair.

Background and notation may be found in [**3**, **8**, and **9**].

Information on vX, the Hewitt realcompactification or Hewitt-Nachbin realcompactification of X, may be found in [**3** and **9**]. However, the following information from [**3**] will be useful.

DEFINITION 1 [**3**]. A z-ultrafilter is real if it has the countable intersection property. A z-ultrafilter that is not real is hyperreal. A space is realcompact iff every real z-ultrafilter is fixed. A z-filter is free or fixed if the intersection of all its members is empty or nonempty, respectively.

We assume all spaces are Tychonoff.

2. The extension αX.

DEFINITION 2. For any Tychonoff space, $\alpha X = (\beta X - vX) \cup X = \beta X - (vX - X)$.

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THEOREM 1. *αX is pseudocompact.*

PROOF. We show that every nonempty zero set of βX meets αX, since a space S is pseudocompact iff every zero set of βS meets S [**3**, p. 95]. Suppose there exists a zero set $Z \subset \beta(\alpha X)$ such that $Z \cap \alpha X = \emptyset$. Then $Z \subset \upsilon X - X$. So Z is a zero set in υX such that $Z \cap X = \emptyset$, a contradiction, since every zero set in υX meets X. Thus every zero set of $\beta(\alpha X)$ meets αX and hence αX is pseudocompact by [**3**, p. 95].

We note that a space X is realcompact iff $\alpha X = \beta X$ and X is pseudocompact iff $\alpha X = X$.

In the next theorem and some subsequent theorems, we will be using the result of Mrowka [**7**] that $\beta X - \upsilon X$ is the union of zero sets of βX. See also [**9**, p. 81].

THEOREM 2. *If A is a subset of $\alpha X - X$ of cardinality less than 2^c, then $\alpha X - A$ is pseudocompact.*

PROOF. If $\alpha X - A$ is not pseudocompact, there exists a zero set H of βX such that $H \cap (\alpha X - A) = \emptyset$ [**3**, p. 95]. But $H \cap \alpha X = \emptyset$ and $\alpha X - X = \beta X - \upsilon X = \bigcup Z_a$ where each Z_a is a zero set of βX. Since $A \subset \bigcup Z_a$, there exists a such that $Z_a \cap H \neq \emptyset$. Since $Z_a \cap H \subset A$, $Z_a \cap H$ is a zero set of βX contained in $\beta X - \upsilon X$ of cardinality strictly less than 2^c which is impossible. See [**3**, p. 133].

COROLLARY 2. *If X is not pseudocompact, then $|\alpha X - X| \geq 2^c$.*

From Example 1 in §7, not all pseudocompactifications of X will have the property in Corollary 2.

It is clear from the preceding theorem that unless X is pseudocompact, there are numerous pseudocompactifications strictly contained in αX and containing X. Furthermore, any set Y such that $\alpha X \subset Y \subset \beta X$ is also pseudocompact. The fact that $p \in \alpha X - X$ implies that $\alpha X - \{p\}$ is pseudocompact contrasts with the fact that, for $p \in \upsilon X - X$, $\upsilon X - \{p\}$ is not realcompact. The difference in behavior is due to the fact that a real z-ultrafilter on being expanded to a larger space is still real, whereas a hyperreal z-ultrafilter's expansion in the larger space may be real.

The next two theorems distinguish αX from other pseudocompactifications of X.

THEOREM 3. *The pseudocompactification αX is the smallest pseudocompactification Y of X contained in βX such that every free hyperreal z-ultrafilter on X converges in Y.*

PROOF. Let Y be a pseudocompactification of X contained in βX but not containing αX. Then there exists some $p \in \alpha X - X$ such that $p \notin Y$. Since $p \in \alpha X - X$ there exists some free hyperreal z-ultrafilter on X which converges to p. This hyperreal filter does not converge in Y.

THEOREM 4. *The pseudocompactification αX is the largest pseudocompactification Y of X, such that $Y \subset \beta X$ and every point in $Y - X$ is contained in a zero set of Y which does not intersect X.*

PROOF. Since $\alpha X - X = \beta X - \upsilon X$, $\alpha X - X$ is the union of zero sets of βX. Let $Y \supset \alpha X$ such that $p \in Y - \alpha X$. Then $p \in \upsilon X - X$ and there exists a zero set of Y, Z, $Z \subset Y - X$. Then $Z \cap \upsilon X$ is a nonempty zero set of $\upsilon X - X$, contradicting the C-embedding of X in υX; so $Y - \alpha X = \emptyset$ and $Y \subset \alpha X$.

THEOREM 5. *Let $X \subset Y \subset \beta X$. Then Y is pseudocompact iff every nonempty zero set of αX intersects Y.*

PROOF. Suppose Y is pseudocompact and Z is a nonempty zero set of αX. If $Z \cap X = \emptyset$, then $Z \cap Y \neq \emptyset$. So let Z be contained in $\alpha X - X$. There exists a zero set H of βX contained in $\alpha X - X = \beta X - vX$ such that $H \cap Z$ is nonempty. Then $H \cap Z$ is a zero set of βX. If $Z \cap Y = \emptyset$, then $(Z \cap H) \cap Y = \emptyset$ contrary to Y being pseudocompact. Conversely, suppose every nonempty zero set of αX meets Y and let Z be a nonempty zero set of βX; then Z intersects αX and $Z \cap \alpha X$ is a zero set of αX and hence meets Y; so $Z \cap Y \neq \emptyset$, and Y is pseudocompact.

3. Local compactness. Since $\alpha X - X = \beta X - vX$ we would expect certain theorems to carry over to $\alpha X - X$ involving $\beta X - X$ when X is realcompact.

THEOREM 6. *If X is locally compact, then each zero set of βX contained in $\alpha X - X$ is a regular closed set of $\alpha X - X$.*

The proof is similar to that of Proposition 4.21 in [**7**].

THEOREM 7. *If $X \cup A \subset \alpha X$ is pseudocompact where $A \subset \alpha X - X$, then A is dense in $\alpha X - X$. The converse is true if X is locally compact.*

This follows from Exercise 8A in [**8**] since $\alpha X - X = \beta X - vX$.

4. α-embedding. Blair studied v-embedding in [**1**]. A useful discussion of this type of embedding may be found in [**9**].

DEFINITION 3. A set $S \subset X$ is v-embedded in X if $vS \subset vX$ (i.e. if $\tau : vS \to vX$ is the Hewitt extension of the inclusion $S \subset X$ then τ induces a homeomorphism from vS onto $\tau(vS)$.

We note the following

LEMMA 8. *A set $S \subset X$ is v-embedded in X iff $vS \subset \beta X$.*

PROOF. v-embedding implies $vS \subset \beta X$ is immediate. Let $vS \subset \beta X$. Since $vX \subset \beta X$, $S \subset vX$ and $vS \cap vX$ is realcompact $vS = vS \cap vX$; so $vS \subset vX$.

THEOREM 8. *If $\alpha S \subset \beta X$ and S is v-embedded in X, then S is C^*-embedded in X.*

PROOF. $\beta S = \alpha S \cup vS \subset \beta X \cup \beta X = \beta X$. So $\beta S \subset \beta X$ and hence S is C^*-embedded in X.

Motivated by Theorem 8 we make the following definition.

DEFINITION 4. A set $S \subset X$ is α-embedded in X if $\alpha S \subset \beta X$.

It is easily seen that if S is C^*-embedded in X, then S is α-embedded in X.

COROLLARY 8. *S is C^*-embedded in X iff S is α-embedded and v-embedded in X.*

However, C^*-embedding of S in X does not imply that $\alpha S \subset \alpha X$. On the other hand, C-embedding of S in X implies $\alpha S \subset \alpha X$. However, S C^*-embedded in X and $\alpha S \subset \alpha X$ does not imply S is C-embedded in X.

One property lacking in an α-embedding is the transitive property.

In the following definition an A-embedding is some specific type of embedding.

DEFINITION 5. An A-embedding is transitive (metatransitive) [hereditary] if S A-embedded in X and X A-embedded in Y imply S is A-embedded in Y (if for $S \subset X \subset Y$, S dense in Y, S A-embedded in X and Y imply X is A-embedded in Y) [if S is dense and A-embedded in Y imply S is A-embedded in X for $S \subset X \subset Y$].

We note that C- and C^*-embeddings have all these properties but v- and z-embeddings are not metatransitive (see Example 5) but are transitive and hereditary whereas α-embeddings are metatransitive and hereditary but not transitive.

THEOREM 9. *An α-embedding is hereditary and metatransitive.*

We will first need the following lemmas, which will be useful in other contexts.

LEMMA 9A. *Let S be dense and C^*-embedded in X, then $\alpha X - \alpha S \subset X - S$.*

PROOF. $\alpha X - X = \beta X - vX \subset \beta X - vS = \beta S - vS = \alpha S - S$. So if $z \in \alpha X - \alpha S$, $z \in X - S$. For if $z \notin X$, $z \in \alpha S - S$. So $z \in X - S$.

LEMMA 9B. *If S is dense and α-embedded in X, then $\alpha X - \alpha S \subset X - S$, or equivalently $\alpha X - X \subset \alpha S - S$.*

PROOF. Let $f\colon \beta S \to \beta X$ be the continuous extension of the identity map of $\alpha S \to \alpha S$. Then $\alpha(f^{-1}(X)) - \alpha S \subset f^{-1}(X) - S$. Also $f(vf^{-1}(X)) \subset vX$ so that $f(\alpha(f^{-1}(X))) \supset \alpha X$. So $\alpha X - \alpha S \subset X - S$.

PROOF OF THEOREM 9. Let P be such that $S \subset P \subset X$, where $\alpha S \subset \beta X$. We will first show that $\alpha S \subset \beta P$. Let f and g be maps such that $\beta S \xrightarrow{f} \beta P \xrightarrow{g} \beta X$, where f is the continuous extension of the identity map from S into P, and g is the continuous extension of the identity map from P into X. The composition map gf is such that $gf(\alpha S) = \alpha S$. Then $f(\alpha S) = g^{-1}(\alpha S)$. So $f(\alpha S) = \alpha S$ and $\alpha S \subset \beta P$. So α-embeddings are hereditary. Since $\alpha P - \alpha S \subset P - S$ by Lemma 9B, $\alpha P \subset \alpha S \cup (P - S) \subset \beta X$. So $\alpha P \subset \beta X$. So α-embeddings are metatransitive.

COROLLARY 9A. *Let P be such that $S \subset P \subset X$, S dense and α-embedded in X; then $\alpha P \subset \alpha S \cup X \subset \alpha S \cup \alpha X$.*

PROOF. $\alpha P - \alpha S \subset P - S$. But $P - S \subset X \subset \alpha X$. So $\alpha P \subset \alpha S \cup X \subset \alpha S \cup \alpha X$.

COROLLARY 9B. *Let P be such that $S \subset P \subset X$, $P \subset \alpha S$ with S dense and α-embedded in X. Then $\alpha P \subset \alpha S$.*

PROOF. $\alpha P - \alpha S \subset P - S$ but $P \subset \alpha S$, so $\alpha P \subset \alpha S$.

COROLLARY 9C. *Let $T = \alpha S \cap X$. Then $\alpha S \cap \alpha X \subset \alpha T \subset \alpha S$ if S is dense and α-embedded in X.*

PROOF. The second part follows from Corollary 9B. Suppose $z \in \alpha S \cap \alpha X$ and $z \notin \alpha T$. Then $z \in \alpha X - \alpha T \subset X - T$. So $z \in \alpha S$, $z \in X$; so $z \in T = \alpha S \cap X$. Contradiction $\alpha S \cap \alpha X \subset \alpha T$.

COROLLARY 9D. *Let S be dense and α-embedded in X and $S \subset P \subset R \subset X$; then P is dense and α-embedded in R.*

5. C-embedding, well-embedding, and αX.

DEFINITION 6. A space S is well-embedded [**5**] in a space X if every zero set of X contained in $X - S$ is completely separated from S. We will say that S is fully well-embedded in X if for $P, R \subset X$ where $S \subset P \subset R \subset X$, then P is well-embedded in R. See Added in proof.

Well-embedding is a necessary and sufficient condition for a C^*-embedded [3] (z-embedded [2]) set to be C-embedded. Because of this important role, we are interested in relating it to α-embedding.

THEOREM 10. *The following are equivalent for $S \subset X$, S dense in X.*

(a) *S is C-embedded in X.*

(b) *S is C^*-embedded in X and $\alpha X - \alpha S \supset X - S$.*

(c) *S is v-embedded in X and the following are satisfied: $\alpha X - \alpha S \supset X - S$ and $\alpha S \subset \alpha X$.*

(d) *S is v-embedded in X and the following is satisfied: If $P \subset R$, where $S \subset P \subset R \subset X$ and $R \neq P$, then $\alpha P \subset \alpha R$ and $\alpha P \neq \alpha R$.*

(e) *S is v-embedded in X and $\alpha X - X = \alpha S - S$.*

PROOF. (a)→(b). $\alpha X - \alpha S = [(\beta X - vX) \cup X] - [(\beta X - vX) \cup S] = X - S$.

(a)→(c). We already have $\alpha X - \alpha S = X - S$. Also $\alpha S = (\beta X - vX) \cup S \subset (\beta X - vX) \cup X = \alpha X$.

(a)→(d) is analogous to (a)→(c), $\alpha P \subset \alpha R$, and $\alpha P = (\beta X - vX) \cup P \neq (\beta X - vX) \cup R = \alpha R$.

(a)→(e) from C-embedding of S in X, $\alpha X - X = \beta X - vX = \beta S - vS = \alpha S - S$.

(e)→(a) since $vS \subset vX$, $\beta S - (\alpha S - S) \subset \beta X - (\alpha X - X) = \beta X - (\alpha S - S)$; so $\beta S \subset \beta X$. Hence S is C^*-embedded in X and $\beta S = \beta X$. So $vS = vX$.

(b)→(a) since S is α-embedded in X, $\alpha X - \alpha S \subset X - S$ by Lemma 9B; so $\alpha X - \alpha S = X - S$. Then $X - S \subset vS$; so $vS = vX$ and S is C-embedded in X.

(c)→(b). $\alpha S \subset \alpha X$ implies S is α-embedded in X and hence C^*-embedded in X.

(d)→(a). As in (c)→(b), S is C^*-embedded in X; by the v-embedding of S in X, S is C-embedded in X.

Many of the above results could also follow from the next theorem.

THEOREM 11. *If S is α-embedded in X and $\alpha X - \alpha S = X - S$, S dense in X, then S is fully well-embedded in X.*

PROOF. From [2], a pseudocompact space is well-embedded in every space in which it is embedded. So $\alpha X - \alpha S$ contains no zero sets and $X - S$ then contains no zero sets. So S is well-embedded in X. For $P, R, S \subset P \subset R \subset X$, $\alpha R - \alpha P = R - P$ since $\alpha X - \alpha S = X - S$. Hence $\alpha R - \alpha P \subset R - P$ by the α-embedding of P in R, which follows from Theorem 9. We then use a similar argument to show that P is well-embedded in R; so that S is fully well-embedded in X.

COROLLARY 11. *Let S be dense and α-embedded in X. Let $P \subset R$, where $P \neq R$ and $S \subset P \subset R \subset X$. If $\alpha P \subset \alpha R$ and $\alpha P \neq \alpha R$, then S is fully well-embedded in X.*

THEOREM 12. *If S is dense and α-embedded in X, then S is fully well-embedded in X iff $\alpha X - \alpha S = X - S$.*

PROOF. If $S = X$ the theorem is immediate. We assume $S \neq X$. The if part follows from Theorem 11. Suppose S is fully well-embedded in X and $\alpha X - \alpha S \neq X - S$. By the α-embedding $\alpha X - \alpha S \subset X - S$. Let $T = \alpha S \cap X$. Let $T \neq S$.

Since $T \subset \alpha S$, S is C^*-embedded in T and βT. By the fully well-embedding, S is C-embedded in T and $\alpha T - \alpha S = T - S$. But from Corollary 9C, $\alpha T \subset \alpha S$. Then $T = S$. Thus, $\alpha X - \alpha S = X - S$.

COROLLARY 12. *Let $T = \alpha S \cap X$ and $Q = (X - T) \cup S$. If S is dense and α-embedded in X, then $\alpha X \subset \alpha Q$, Q is C^*-embedded in X, S is C^*-embedded in T, S is fully well-embedded in Q. and T is fully well-embedded in X.*

PROOF. By Thoerem 9, S is α-embedded in Q and T is α-embedded in X. So by Lemma 9A, $\alpha Q - \alpha S \subset Q - S$ and $\alpha X - \alpha T \subset X - T$. Since $X \cap \alpha S = T$, $\alpha Q - \alpha S = Q - S$, establishing that S is fully well-embedded in Q. Since $\alpha T \cap X = T$, $\alpha X - \alpha T = X - T$; so T is fully well-embedded in X. Since $T \subset \beta S$, S is C^*-embedded in T. We now show that $\alpha X \subset \alpha Q$. Suppose $\alpha Q \cap X = M$. Then $Q \subset M \subset X$. Similar to the argument that S is fully well-embedded in Q, we can show that Q is fully well-embedded in $P = Q \cup (X - M)$ since also S is fully well-embedded in Q, Hence, S is fully well-embedded in P. By Theorem 12, $\alpha P - \alpha S = P - S$; but $\alpha P - \alpha S \subset Q - S$ since $T \subset \alpha S$. So $P = Q$ and $\alpha Q \cap X = X$ and $X \subset \alpha Q$ and $\alpha X \subset \alpha Q$ by Corollary 9B. Hence Q is C^*-embedded in X.

6. α-embedding and C^*-embedding.

THEOREM 13. *Let S be dense in X and let $T = \alpha S \cap X$ and $Q = X - (T - S)$. Then the following relations hold:* (a) $\leftrightarrow$ (b) $\rightarrow$ (c) $\leftrightarrow$ (e) $\rightarrow$ (g). (e) $\leftrightarrow$ (g) + (h). (b) $\rightarrow$ (d) $\leftrightarrow$ (f). (g) + (i) $\rightarrow$ (f).

(a) *S is C^*-embedded in X.*

(b) *S is C^*-embedded in X, T is C-embedded in X, and S is C-embedded in Q.*

(c) *There exists P such that $\alpha P \supset \alpha X \cup \alpha S$, where $S \subset P \subset X$.*

(d) *There exists P such that $\alpha P \subset \alpha S \cap \alpha X$, where $S \subset P \subset X$.*

(e) $\alpha Q = \alpha X \cup \alpha S$.

(f) $\alpha T = \alpha X \cap \alpha S$.

(g) *S is α-embedded in X.*

(h) *For $f\colon \beta S \rightarrow \beta X$, where f is the extension of the identity map from $S \rightarrow S$, the image of vS is realcompact.*

(i) *For $f\colon \beta S \rightarrow \beta X$, where f is the extension of the identity map from $S \rightarrow S$, the image of vT is realcompact.*

PROOF. (b)$\rightarrow$(a) is immediate.

(a)$\rightarrow$(b) follows from Corollary 12.

(b)$\rightarrow$(e). Since S is C-embedded in Q from Theorem 10, $\alpha S \subset \alpha Q$; by Theorem 9 and Corollary 12, $\alpha X \subset \alpha Q$, so $\alpha Q \supset \alpha X \cup \alpha S$, and by Corollary 9A, $\alpha Q = \alpha X \cup \alpha S$.

(e)$\rightarrow$(g). Since $\alpha Q \supset \alpha X$, Q is C^*-embedded in X; so $\alpha Q \subset \beta X = \beta Q$ and $\alpha S \subset \alpha Q$. So $\alpha S \subset \beta X$.

(e)$\rightarrow$(h). Since (e)$\rightarrow$(g), $f\colon \beta S \rightarrow \beta X$ maps αS onto αS. If $f(vS) = (\beta Q - \alpha S) \cup Q$ is not realcompact, then $vQ \cap (\alpha S - S) \neq \emptyset$. So $\alpha S \notin \alpha Q$. Hence $f(vS)$ is realcompact.

(g) + (h) $\rightarrow$ (e). $f\colon \beta S \rightarrow \beta X$ maps αS onto αS. If $f(vS) = (\beta Q - \alpha S) \cup Q$ is realcompact, $f(vS) = vQ$ and $vQ \cap (\alpha S - S) = \emptyset$. So $\alpha S \subset \alpha Q$ and $\alpha Q = \alpha S \cup \alpha X$ by Corollary 12.

(e)$\rightarrow$(c) is immediate.

(c)$\to$(e). Suppose $\alpha P \supset \alpha S \cup \alpha X$; then P is C^*-embedded in X; so $\alpha S \subset \beta X$ and hence $\alpha P - \alpha S \subset P - S$. But $X \subset \alpha P$ so $\alpha P - \alpha S = X - T = Q - S$. So $Q \subset P$. Since $X \subset \alpha Q$ by Corollary 12, $\alpha P - \alpha Q = \varnothing$. So $\alpha P \subset \alpha Q$. By Corollary 9A, $\alpha Q = \alpha S \cup \alpha X$.

(b)$\to$(f). Since $T \subset \alpha S$ and $\alpha T - \alpha S \subset T - S$, $\alpha T \subset \alpha S$. Since T is C-embedded in X, $vT = vX$. So

$$\alpha X = (\beta X - vX) \cup X = (\beta X - vX) \cup T \supset (\beta T - vT) \cup T = \alpha T;$$

so $\alpha T \subset \alpha X \cap \alpha S$. By Corollary 9C, $\alpha T = \alpha X \cap \alpha S$.

(f)$\to$(d) is immediate.

(d)$\to$(f). If $\alpha P \subset \alpha S$, $P \subset \alpha S$ so $P \subset T$; hence P is C^*-embedded in T and $\alpha T - \alpha P \subset T - P$. So $\alpha T \subset \alpha P \cup (T - P) \subset \alpha S \cap \alpha X$; hence $\alpha T \subset \alpha X \cap \alpha S$ and, by Corollary 9C, $\alpha T = \alpha X \cap \alpha S$.

(f)$\to$(i). The proof is similar to (e)$\to$(h). (g) + (i) $\to$ (f). The proof is similar to (g) + (h) $\to$ (e).

Example 5 in the next section shows that (f)$\not\to$(g). Also (b) $\not\to$(a) (see Example 7). We note that if S is v-embedded in X, then (h) is satisfied. See Added in proof.

THEOREM 14. *Let S be dense and α-embedded in X. Then the following relations hold. Any two of* (a), (b), *and* (c) *imply the other. And* (c) $\leftrightarrow$ (d) , *where* (a), (b), (c) *and* (d) *are given below. In the following, Q and T are defined as in Theorem* 13:

(a) $\alpha Q = \alpha X \cup \alpha S$,
(b) $\alpha T = \alpha X \cap \alpha S$,
(c) $\alpha Q - \alpha X = \alpha S - \alpha T$,
(d) $vT - vS = vX - vQ$.

We omit the proof.

7. Some examples. In connection with Corollary 2 we have the following.

EXAMPLE 1. Let $\{N_a : a \in A\}$ be the set of countable subsets of N. Let $\mathscr{F}_a$ be a free ultrafilter on N containing N_a. Let $P = N \cup \{\mathscr{F}_a : a \in A\}$. Then $N \subset P \subset \beta N$, $|P| = c$, and P is pseudocompact.

The authors are indebted to Alan Dow for pointing out the above example. See also Mrowka [**6**]. We note that if R is such that $P \subset R \subset \beta N$, then R is pseudocompact so that N has 2^{2^c} pseudocompactifications contained in βN, and hence X has 2^{2^c} pseudocompactifications in βX if X is not pseudocompact. The authors are grateful to the referee for pointing out this result.

EXAMPLE 2. Let S be the rationals and X the reals. Then $\alpha X - \alpha S \supset X - S$ and $\alpha X - \alpha S \neq X - S$. (Compare with Lemma 9B.)

In the next three examples, Q and T are defined as in Theorem 13.

EXAMPLE 3. Let $X = \beta N$ and $S = N$. Then $\alpha S = \alpha X = \beta N$ and $T = \beta N$. But there exists P, $S \subset P \subset T$ such that $\alpha P = P \neq \alpha S \cap \alpha X = \beta N$. In fact, there is no pseudocompact lower bound for αP where $S \subset P \subset T$.

EXAMPLE 4. Let $S = N$, X, the one-point compactification KN of N. Then $\alpha X = KN$ and $\alpha S = \beta N$, $T = N$, $Q = KN$, $\alpha T = \beta N$, and $\alpha Q = KN$; so $\alpha S \not\subset \beta X$ and $\alpha T \neq \alpha S \cap \alpha X$. Furthermore, $\alpha Q - \alpha S = \alpha X - \alpha T = KN - \beta N = KN - N = X - T$. Also $\alpha S \cap \alpha X$ is not pseudocompact.

EXAMPLE 5. Let $S = N$, $X = KN$, where KN is obtained from βN by identifying two points $\{x\}$ and $\{y\}$ of $\beta N - N$. Then $T = \beta N - (\{x\} \cup \{y\})$ and $Q = N \cup \{z\}$ where $\{z\}$ is the point resulting from the identification of $\{x\}$ and $\{y\}$. Then $\alpha S = \beta N$, $\alpha X = KN = \alpha Q$, $\alpha T = T$, $\alpha T = \alpha X \cap \alpha S$ but $\alpha S \notin \beta X$ (see Theorem 13). Furthermore $\alpha Q - \alpha S = \{z\} = \alpha X - \alpha T$ and $\alpha S - \alpha T = \{x\} \cup \{y\}$ and $\alpha X = \alpha Q$.

The above example also shows that v-embedding and z-embedding are not metatransitive. For N is z-embedded and v-embedded in KN but T is not v-embedded or z-embedded in KN. We note that S is α-embedded in T, and T is α-embedded in X, but S is not α-embedded in X.

EXAMPLE 6. Let Y be the one-point compactification of an infinite discrete space X of nonmeasurable cardinality. Then X is v-embedded and fully well-embedded in Y but not α-embedded in Y.

EXAMPLE 7. Let S be pseudocompact with a compactification $X = KS \neq \beta S$. Then $Q = X$, so $\alpha Q = \alpha S \cup \alpha X$ and S is not C^*-embedded in X. (See Theorem 13.)

ADDED IN PROOF. It can be proved that well-embedding is equivalent to full well-embedding for dense embeddings and in Theorem 13.

(g) $\rightarrow$ (h) and (g) $\rightarrow$ (i) so
(g) $\rightarrow$ (e) and (g) $\rightarrow$ (f).

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