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A RIBBON KNOT GROUP WHICH HAS NO FREE BASE

KATSUYUKI YOSHIKAWA

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ABSTRACT. We consider the following problem: If a group G satisfies the conditions (1) G has a finite presentation with $r+1$ generators and r relators, and (2) there exists an element x of G such that $G = \langle\langle x \rangle\rangle^G$ where $\langle\langle x \rangle\rangle^G$ is the normal closure of x in G, then is G an HNN (Higman-Neumann-Neumann) extension of a free group of finite rank? In this paper, we give a negative answer to the problem. Thus it follows that there exists a ribbon n-knot group ($n \geq 2$) which has no free base.

1. Introduction. Let G be a 2-knot group, i.e., the group of some 2-sphere smoothly embedded in the 4-sphere S^4. Then it is easily seen that if G is an HNN extension of some free group of finite rank, then G has deficiency one. Conversely, the following conjecture is raised:

CONJECTURE. *If the deficiency of G is one, then G has a free base of finite rank.*

The purpose of this paper is to give counterexamples to this conjecture, that is,

THEOREM 1. *Let G be a group presented by*

$$\langle a, b, t \colon a^p = b^q, ta^\alpha t^{-1} = b^\beta \rangle, \tag{1.1}$$

where p, q, α, β are nonzero integers such that $p\beta - q\alpha = \pm 1$ and $p, q, \alpha, \beta \neq \pm 1$. Then,

(1) G is a ribbon 2-knot group, i.e., a 2-knot group with Wirtinger presentation of deficiency one, and

(2) G has no free base of finite rank.

Thus, our examples show that Gutierrez's theorem [**2**, p. 287, Theorem (iii)] is false. In §4, we prove that any 2-knot group with one-relator Wirtinger presentation has a free base of finite rank.

2. Preliminaries. Let $\{A_i, \theta_{jk}\}$ be a collection of groups A_i and isomorphisms $\theta_{jk} \colon U_{jk} \to U_{kj}$ associated with certain pairs of A_j, A_k such that $\theta_{jk} = \theta_{kj}^{-1}$, where U_{jk} and U_{kj} are subgroups of A_j and A_k respectively. With the collection $\{A_i, \theta_{jk}\}$ we associate a linear graph each of whose vertices corresponds to a group A_i and each of whose edges joins two vertices A_j and A_k if there exists θ_{jk} (and hence θ_{kj}).

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If this graph is a tree, the group defined by

$$\left\langle \prod_i {}^* A_i \colon U_{jk} = \theta_{jk}(U_{jk}) \text{ for all edges } e_{jk} \right\rangle$$

is called a *tree product of the factors* A_i (*with the subgroups* U_{jk} *and* U_{kj} *amalgamated under* θ_{jk}) [**4**].

An HNN extension with more than one stable letter is defined as follows [**4**, **5**]: Let H be a group and let $\{A_1, \ldots, A_n\}$, $\{B_1, \ldots, B_n\}$ be sets of subgroups of H with isomorphisms $\phi_i \colon A_i \to B_i$, $i = 1, \ldots, n$. Then the *HNN extension of the base* H *with stable letters* $t_1, \ldots, t_n$ *and associated subgroups* $\{A_1, \ldots, A_n\}$ and $\{B_1, \ldots, B_n\}$ is the group given by

$$\langle H, t_1, \ldots, t_n \colon t_i a_i t_i^{-1} = \phi_i(a_i),\ a_i \in A_i,\ i = 1, \ldots, n \rangle.$$

Finally, we give some lemmas. Let G be as in Theorem 1, and let $H = gp\{a, b\} = \langle a, b \colon a^p = b^q \rangle$.

LEMMA 2.1. *The element b is not conjugate to $a^{\alpha k}$ or $b^{\beta k}$ in H for any integer k. Similarly, a is not conjugate to $a^{\alpha k}$ or $b^{\beta k}$.*

PROOF. Let f be the homomorphism of H onto the infinite cyclic group $Z = \langle z : \rangle$ defined by $f(a) = z^q$ and $f(b) = z^p$. Then, since $p\beta - q\alpha = \pm 1$ and $p, q, \alpha, \beta \neq \pm 1$, it follows that $f(a^{\alpha k})$ and $f(b^{\beta k})$ are different from z^p and z^q. Therefore the lemma holds.

LEMMA 2.2. *For any $S \in G$, a does not commute SbS^{-1} in G.*

PROOF. Let $S = g_0 t^{\varepsilon_1} g_1 \cdots t^{\varepsilon_n} g_n$ be reduced, that is, $\varepsilon_i = \pm 1$, $g_i \in H$, and there is no subword $t g_i t^{-1}$ with $g_i \in gp\{a^\alpha\}$ or $t^{-1} g_i t$ with $g_i \in gp\{b^\beta\}$ (cf. [**5**]). Then, we have

$$\begin{aligned} &a(SbS^{-1})a^{-1}(Sb^{-1}S^{-1}) \\ &\quad = a(g_0 t^{\varepsilon_1} \cdots t^{\varepsilon_n} g_n) b (g_0 t^{\varepsilon_1} \cdots t^{\varepsilon_n} g_n)^{-1} \\ &\qquad \cdot a^{-1} (g_0 t^{\varepsilon_1} \cdots t^{\varepsilon_n} g_n) b^{-1} (g_0 t^{\varepsilon_1} \cdots t^{\varepsilon_n} g_n)^{-1}. \end{aligned} \tag{2.1}$$

By Lemma 2.1, we can see that

$$g_n b^{\pm 1} g_n^{-1} \notin gp\{a^\alpha\} \cup gp\{b^\beta\},$$

and

$$g_0^{-1} a^{-1} g_0 \notin gp\{a^\alpha\} \cup gp\{b^\beta\}.$$

Therefore, (2.1) is reduced. If $n \geq 1$, then it follows from Britton's lemma [**5**] that equation (2.1) does not define the identity element. When $n = 0$, (2.1) yields $a g_0 b g_0^{-1} a^{-1} g_0 b^{-1} g_0^{-1}$. Then, from [**6**, Theorem 4.5], this is not the identity element in H, and so in G. Thus we complete the proof.

LEMMA 2.3. *Suppose that H is also described as a tree product $T = \{A_i, \theta_{jk}\}$ of the infinite cyclic groups A_i. Then, H is a tree product of two vertices, say A_1, A_2, of $\{A_i\}$ and the edge is given by $a_1^p \to a_2^q$, where a_i is a generator of A_i, $i = 1, 2$.*

PROOF. Since H is finitely generated, we may assume that the number of the vertices $\{A_i\}$ is finite and the tree product T is proper, i.e., each of the amalgamated subgroups is properly contained in the adjoining vertices [**4**, p. 237].

Let H^* be the normal closure of the nonextremal vertices of $\{A_i\}$ in H. Then, since T is proper, the factor group H/H^* is a free product of nontrivial cyclic groups and the rank (i.e., miminum number of generators) of H/H^* is equal to the number of extremal vertices of $\{A_i\}$. Since the rank of H is 2, it follows that T has exactly two extremal vertices (i.e., T is a stem product). Thus, from [**9**, p. 97, Theorem 2] and [**7**, p. 320, Theorem 1], we obtain the lemma.

LEMMA 2.4. *The induced presentation* $\langle a, b, t \colon a^p = b^q, ta^\alpha t^{-1} = b^\beta, t\rangle$ *of* (1.1) *is AC-equivalent to the trivial presentation* $\langle : \rangle$.

PROOF. The lemma is an immediate consequence of [**1**, Theorem 2.1].

3. Proof of Theorem 1. From Lemma 2.4 and [**13**], the first assertion holds. We proceed to the proof of the second assertion. We will prove only the case of $p = 2$, $q = 3$, $\alpha = 3$, $\beta = 4$, and the other cases can be shown by the same arguments.

Assume that G is an HNN extension of a free group F of finite rank d with a single stable letter t. Then the commutator subgroup G' of G is the amalgamated free product of infinitely many factors

$$\cdots *_{F_{-10}} F_0 *_{F_{01}} F_1 *_{F_{12}} F_2 * \cdots , \tag{3.1}$$

where F_i are copies of F. Let $F(i,j)$ denote the subgroup $F_i *_{F_{i,i+1}} \cdots *_{F_{j-1,j}} F_j$, $i \leq j$. On the other hand, from (1.1), G' is also the amalgamated free product

$$\cdots *_{H_{-10}} H_0 *_{H_{01}} H_1 *_{H_{12}} H_2 * \cdots , \tag{3.2}$$

where $H_i = \langle a_i, b_i \colon a_i^2 = b_i^3 \rangle$, $H_{i,i+1} = gp\{a_{i+1}^3\} = gp\{b_i^4\}$, $a_i = t^i a t^{-i}$ and $b_i = t^i b t^{-i}$; each amalgamation is given by mapping $a_{i+1}^3 \to b_i^4$. Let $H(i,j)$ denote the subgroup $H_i *_{H_{i,i+1}} \cdots *_{H_{j-1,j}} H_j$, $i \leq j$.

Now, a_0^2 is an element of the commutator subgroup G'. Hence, by (3.1), there exist integers r, s $(r \leq s)$ such that $a_0^2 \in F(r,s)$. Moreover, since $F(r,s)$ is finitely generated, it follows from (3.2) that there exist integers n, m $(n \leq m)$ such that $F(r,s) \subset H(n,m)$. Here we may assume that $n \leq 0 < m$.

The center $C(H_i)$ of H_i is the infinite cyclic group generated by a_i^2 $(= b_i^3)$. Therefore, from [**6**, p. 211], we have

$$\begin{aligned} C(H(n, n+1)) &= gp\{b_n^3\} \cap gp\{a_{n+1}^2\} \cap H_{n,n+1} \\ &= gp\{(b_n^3)^8\} = gp\{(a_n^2)^8\}. \end{aligned}$$

Similarly, we can obtain

$$C(H(n,m)) = gp\{(a_n^2)^{8^{m-n}}\},$$

and it is infinite cyclic. Since $(a_0^2)^9 = (a_{-1}^2)^8$, we can see that

$$\{(a_0^2)^{9^{-n}}\}^{8^m} = \{(a_n^2)^{8^{-n}}\}^{8^m} = (a_n^2)^{8^{m-n}} \in C(H(n,m)).$$

Thus, since $\{(a_0^2)^{9^{-n}}\}^{8^m} \in F(r,s) \subset H(n,m)$, it follows that $\{(a_0^2)^{9^{-n}}\}^{8^m}$ $(\neq 1)$ is in the center $C(F(r,s))$. Hence $C(F(r,s))$ is nontrivial. Thus, from [**6**, p. 211], the free group F_r must have the nontrivial center. Therefore, we have $d = 1$, and G is presented by $\langle c, x \colon xc^k x^{-1} = c^l \rangle$. Now the Alexander polynomial $\Delta(t)$ of G is $9t - 8$. Hence we get $k = \pm 9$, $l = \pm 8$ or $k = \pm 8$, $l = \pm 9$. Without loss of generality, we may assume that $k = 9$, $l = 8$.

Thus, if G is an HNN extension of a free group of finite rank, then G must be isomorphic to the group G^* presented by

$$\langle c, x\colon xc^9x^{-1} = c^8\rangle.$$

To complete the proof, we will show the following:

CLAIM. *G cannot be isomorphic to G^*.*

We assume that there exists an isomorphism Φ of G onto G^*. Let $H = gp\{a, b\}$ in G. We consider G^* as an HNN extension of F $(= \langle c\,:\rangle)$ with associated subgroups $gp\{c^9\}$ and $gp\{c^8\}$. Using the subgroup theorem for HNN extensions in [**3**], we can describe the subgroup $\Phi(H)$ of G^* as follows:

$\Phi(H)$ is an HNN extension with stable letters $t_1, \dots, t_n$ $(n \geq 0)$ whose base is a tree product of vertices $dFd^{-1} \cap \Phi(H)$ where d ranges over a double coset representative system for G^* mod $(\Phi(H), F)$; the amalgamated and associated subgroups are contained in vertices of this base.

We can see the following:

(1) $dFd^{-1} \cap \Phi(H) \cong 1$ or Z.

(2) $n = 0$.

The first assertion follows immediately from the fact that $F \cong Z$. Since $H_1(\Phi(H)) \cong Z$, it follows that n is at most one. If $n = 1$, then its associated subgroups must be isomorphic to 1 or Z. However, this is impossible for the group $\Phi(H)$. Hence, we conclude that $n = 0$.

Thus, $\Phi(H)$ is a tree product of infinite cyclic groups $dFd^{-1} \cap \Phi(H)$. Therefore, from Lemma 2.3, $\Phi(H)$ is an amalgamated product of two vertices $d_1F\,d_1^{-1} \cap \Phi(H)$ and $d_2F\,d_2^{-1} \cap \Phi(H)$ with subgroups $gp\{u_1^2\}$ and $gp\{u_2^3\}$ amalgamated under $u_1^2 \to u_2^3$, where u_i $(i = 1, 2)$ is a generator of $d_iF\,d_i^{-1} \cap \Phi(H)$. Thus, we have

$$\Phi(H) = \langle u_1, u_2\colon u_1^2 = u_2^3\rangle.$$

Hence, there exists an automorphism f of $\Phi(H)$ such that $\Phi(a) = f(u_1)$ and $\Phi(b) = f(u_2)$. From [**12**], the automorphism f of $\Phi(H)$ is given by the form

$$f(u_1) = Wu_1^{\varepsilon}W^{-1}, \qquad f(u_2) = Wu_2^{\varepsilon}W^{-1},$$

where $W \in \Phi(H)$ and $\varepsilon = \pm 1$. Thus we have

$$\Phi(a) = Wu_1^{\varepsilon}W^{-1}, \qquad \Phi(b) = Wu_2^{\varepsilon}W^{-1}.$$

Now, since $F = \langle c\,:\rangle$, it follows that $u_i = d_ic^{\nu_i}\,d_i^{-1}$, $i = 1, 2$, where ν_i are nonzero integers. Hence we can easily see that

$$u_1(d_1\,d_2^{-1}u_2\,d_2\,d_1^{-1})u_1^{-1} = d_1\,d_2^{-1}u_2\,d_2\,d_1^{-1}.$$

Thus, we obtain

$$\begin{aligned} 1 &= \Phi^{-1}(Wu_1\,d_1\,d_2^{-1}u_2\,d_2\,d_1^{-1}u_1^{-1}\,d_1\,d_2^{-1}u_2^{-1}\,d_2\,d_1^{-1}W^{-1}) \\ &= a^{\varepsilon}\Phi^{-1}(S)b^{\varepsilon}\Phi^{-1}(S^{-1})a^{-\varepsilon}\Phi^{-1}(S)b^{-\varepsilon}\Phi^{-1}(S^{-1}), \end{aligned}$$

where $S = Wd_1\,d_2^{-1}W^{-1}$. Therefore, there exists $\tilde{S} \in G$ such that

$$a(\tilde{S}b\tilde{S}^{-1})a^{-1}(\tilde{S}b^{-1}\tilde{S}^{-1}) = 1.$$

However, this contradicts Lemma 2.2. Hence, there cannot exist an isomorphism of G onto G^*. Thus, we complete the proof of the claim, and therefore Theorem 1.

REMARKS. (1) It can be shown that if an n-knot group G is an HNN extension of a free group of infinite rank, then G has a free base of finite rank (cf. [**14**]). Therefore, the groups in Theorem 1 have no *free* base.

(2) Each group in Theorem 1 is a ribbon n-knot group ($n \geq 2$) but not a 1-knot group because its Alexander polynomial has degree one. L. Neuwirth [**8**] proved that if the commutator subgroup of a 1-knot group is finitely generated, then it is free of finite rank. It still remains open whether there exists a 1-knot group which has no free base.

4. Knot groups with one-relator Wirtinger presentation. In this section, we willl show that

THEOREM 2. *Any 2-knot group with one-relator Wirtinger presentation has a free base of finite rank.*

PROOF. Let G be a group presented by

$$\langle x, y\colon y = W(x,y)xW(x,y)^{-1}\rangle, \tag{4.1}$$

where $W(x,y)$ is a reduced word ($\neq 1$) which does not begin in y or y^{-1} and does not end in x or x^{-1}. Setting $a = yx^{-1}$ and deleting y in (4.1), we obtain

$$\begin{aligned} G &= \langle x, a\colon ax = W(x,ax)xW(x,ax)^{-1}\rangle \\ &= \langle x, a\colon xW'x^{-1} = a^{-1}W'\rangle, \end{aligned} \tag{4.2}$$

where W' is the reduced word of $W(x, ax)$. We notice that

$$W' \text{ begins in neither } a \text{ nor } x^{-1}a^{-1}. \tag{$*$}$$

Let $\tilde{W}(a_i)$ be the word obtained by rewriting $W'x^{-k}$, where k is the exponent sum of W' on x, in terms of $a_i = x^i a x^{-i}$. Let m be the minimum and M the maximum subscript j such that a_j occurs in $\tilde{W}(a_i)$. If $M \geq 0$, then we rewrite the relation in (4.2) as follows:

$$x\tilde{W}(a_i)x^{-1} = a_0^{-1}\tilde{W}(a_i).$$

If $M < 0$, then we have

$$x(axW'x^{-1}W'^{-1})x^{-1} = xa_0\tilde{W}(a_{i+1})x^{-1}\tilde{W}(a_{i+1})^{-1} = 1.$$

Thus, we obtain

$$G = \langle a_\delta, a_{\delta+1}, \ldots, a_D, x\colon xa_\delta x^{-1} = a_{\delta+1}, \ldots, xa_{D-1}x^{-1} = a_D, xPx^{-1} = Q\rangle,$$

where $D = \max\{M, 0\}$, and δ, P, Q are given by

(1) if $M \geq 0$, then $\delta = \min\{m, 0\}$, $P = \tilde{W}(a_i)$, $Q = a_0^{-1}\tilde{W}(a_i)$,

(2) if $M < 0$, then $\delta = m + 1$, $P = a_0\tilde{W}(a_{i+1})$, $Q = \tilde{W}(a_{i+1})$.

From ($*$), $\tilde{W}(a_i)$ does not begin in a_0, and $\tilde{W}(a_{i+1})$ does not begin in a_0^{-1}. Therefore, in both cases of (1), (2), P involves a_D and Q involves a_δ. Consequently, $\{a_\delta, \ldots, a_{D-1}, P\}$ and $\{a_{\delta+1}, \ldots, a_D, Q\}$ freely generate free subgroups of rank $D - \delta + 1$ in the free group F on $a_\delta, \ldots, a_D$, respectively. It follows that F is a free base of G. Thus we obtain the theorem.

REMARKS. (1) There exists a "one-relator" 2-knot group which has no free base. For example, in the case of $p = 2$, $q = 3$, $\alpha = 3$, $\beta = 4$, the group given by (1.1) is a one-relator group.

(2) In [**11**, p. 125], E. S. Rapaport showed that for any one-relator 2-knot group G, its commutator subgroup G' is finitely generated if and only if G' is free.

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FACULTY OF SCIENCE, KWANSEI GAKUIN UNIVERSITY, NISHINOMIYA, HYOGO 662, JAPAN