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THE GAP BETWEEN cmp X AND def X CAN BE ARBITRARILY LARGE

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Dedicated to Professor Yukihiro Kodama on his 60th birthday

ABSTRACT. We give an example of a separable metrizable space X with $\operatorname{def} X - \operatorname{cmp} X = n$ for every $n \in \mathbb{N}$.

1. Introduction. In this paper all spaces are separable and metrizable.

The *compactness degree*, $\operatorname{cmp} X$, of a space X is defined as follows: a space X satisfies $\operatorname{cmp} X = -1$ if X is compact; if n is a nonnegative integer, then $\operatorname{cmp} X \leq n$ means that each point of X has arbitrarily small neighborhoods U with $\operatorname{cmp} \operatorname{Bd} U \leq n-1$. We put $\operatorname{cmp} X = n$ if $\operatorname{cmp} X \leq n$ and $\operatorname{cmp} X \nleq n-1$. If there is no integer n for which $\operatorname{cmp} X \leq n$, then we put $\operatorname{cmp} X = \infty$.

The *compactness deficiency*, $\operatorname{def} X$, of a space X is the least integer n for which X has a compactification αX with $\dim(\alpha X - X) \leq n$. We allow n to be ∞.

In general, the inequality $\operatorname{cmp} X \leq \operatorname{def} X$ holds. The well-known conjecture of J. de Groot (see [**2**]) that $\operatorname{cmp} X = \operatorname{def} X$ has been negatively solved by R. Pol [**5**]; the space X of Pol's example has $\operatorname{cmp} X = 1$ and $\operatorname{def} X = 2$. In the review of R. Pol's paper [**5**], J. van Mill [**3**] states "It seems still to be open whether the gap between $\operatorname{cmp} X$ and $\operatorname{def} X$ can be arbitrarily large."

The purpose of this paper is to answer this question affirmatively. Namely, we shall give the following example.

EXAMPLE. For every $n \in \mathbb{N}$ there exists a space X such that $\operatorname{def} X - \operatorname{cmp} X = n$.

2. Preliminaries. Let $\mathcal{S}$ be a collection of subsets of a space X. Then we shall write $[\mathcal{S}]^n$ for $\{\mathcal{T}: \mathcal{T} \subset \mathcal{S} \text{ with } |\mathcal{T}| = n\}$, $\operatorname{Bd} \mathcal{S}$ for $\{\operatorname{Bd} S: S \in \mathcal{S}\}$ and $\bigcap \mathcal{S}$ for $\bigcap\{S: S \in \mathcal{S}\}$.

Let Y be a subspace of a space X and $\mathcal{U}$ a collection of open subsets of X. Then $\mathcal{U}$ is an *outer base* for Y in X if for every $y \in Y$ and any neighborhood V of y in X there is $U \in \mathcal{U}$ such that $y \in U \subset V$.

The following lemma is needed in §4; the proof is straightforward.

2.1. LEMMA. *Let X be a space with $\operatorname{def} X < n$ and $\{(E_j, F_j): 1 \leq j \leq n\}$ a collection of pairs of disjoint compact subsets of X. Then for each j, $1 \leq j \leq n$, there is a partition T_j in X between E_j and F_j such that $\bigcap\{T_j: 1 \leq j \leq n\}$ is compact.*

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To show our example it suffices to construct a space Y with $n \leq \operatorname{def} Y - \operatorname{cmp} Y < \infty$. Indeed, for this space Y we construct another space Z with $\operatorname{cmp} Z = \operatorname{def} Z = \operatorname{def} Y - n$; such a space exists (see [**2**, Theorem 3.1.1]). Let $X = Y \oplus Z$ be the topological sum of Y and Z. Then

$$\operatorname{cmp} X = \max\{\operatorname{cmp} Y, \operatorname{cmp} Z\} = \operatorname{cmp} Z = \operatorname{def} Y - n$$

and

$$\operatorname{def} X = \max\{\operatorname{def} Y, \operatorname{def} Z\} = \operatorname{def} Y.$$

Thus we have $\operatorname{def} X - \operatorname{cmp} X = n$.

In the next section we shall construct a space X such that $m \leq \operatorname{def} X - \operatorname{cmp} X \leq 2m$ for every $m \in \mathbb{N}$.

Throughout the rest of this paper, we shall fix a positive integer m and put $n = 2m + 1$. Let $I = [0, 1]$ be the closed unit interval.

3. Construction. Let

$$\partial I^n = \{(x_j) \in I^n : x_j = 0 \text{ or } 1 \text{ for some } j,\ 1 \leq j \leq n\}$$

be the combinatorial boundary of the n-dimensional cube I^n. We take countable, dense subsets D_0 and D_1 in $(0,1)$ with $D_0 \cap D_1 = \varnothing$. Let us set

$$M_i = \{(x_j) \in (0,1)^n : |\{j : x_j \in D_i\}| \geq n - m\},$$

and

$$L_i = (0,1)^n - M_i$$

for each $i = 0, 1$. Then, obviously, $M_0 \cap M_1 = \varnothing$ and by [**1**, 1.8.5], $\dim L_i = m$. Then, by [**4**, 12.12–13], there are two collections $\mathcal{B}_0$ and $\mathcal{B}_1$ of open subsets of I^n satisfying the following conditions (1) to (6) below:

(1) $\mathcal{B}_0$ is an outer base for $(I^{n-1} \times [0, \frac{2}{3})) \cap \partial I^n$ in I^n,
(2) $\mathcal{B}_1$ is an outer base for $(I^{n-1} \times (\frac{1}{3}, 1]) \cap \partial I^n$ in I^n,
(3) $\bigcap \mathcal{F} \cap L_i = \varnothing$ for every $\mathcal{F} \in [\operatorname{Bd} \mathcal{B}_i]^{m+1}$ and each $i = 0, 1$,
(4) $\operatorname{Cl} B \subset I^{n-1} \times [0, \frac{2}{3})$ for every $B \in \mathcal{B}_0$,
(5) $\operatorname{Cl} B \subset I^{n-1} \times (\frac{1}{3}, 1]$ for every $B \in \mathcal{B}_1$, and
(6) $|\mathcal{B}_i| = \omega$ for each $i = 0, 1$.

By (6), $[\operatorname{Bd} \mathcal{B}_i]^{m+1}$ is countable; therefore, we enumerate it as $[\operatorname{Bd} \mathcal{B}_i]^{m+1} = \{\mathcal{F}_{ij} : j \in \mathbb{N}\}$. Let us see $F_{ij} = \bigcap \mathcal{F}_{ij}$, and let

$$E_{ik} = \bigcup\{F_{ij} : j \leq k\} - \partial I^n$$

for $i = 0, 1$ and $k \in \mathbb{N}$. Then, by (3), we have $E_{0k} \cap E_{1k} \subset M_0 \cap M_1 = \varnothing$. Thus E_{0k} and E_{1k} are disjoint closed subsets of $(0,1)^n$, therefore we can take disjoint open subsets U_{0k} and U_{1k} in $(0,1)^n$ such that

(7) $E_{ik} \subset U_{ik}$ for each $i = 0, 1$,
(8) $U_{0k} \subset I^{n-1} \times [0, \frac{2}{3})$, and
(9) $U_{1k} \subset I^{n-1} \times (\frac{1}{3}, 1]$.

Let us set

$$X_k = (I^n - U_{0k} \cup U_{1k}) \times \{1/k\} \text{ for every } k \in \mathbb{N},$$
$$X_0 = \partial I^n \times \{0\}, \text{ and}$$
$$X = \bigcup\{X_k : k = 0, 1, 2, \dots\}.$$

We regard X as the subspace of the $(n+1)$-dimensional cube

$$I^{n+1} = \Pi\{I_j : 1 \leq j \leq n+1\},$$

where I_j is the copy of I.

4. $2m \leq \operatorname{def} X \leq 2m+1$**.** Note that $\operatorname{def} Y \leq \dim Y$ for every space Y (see [**5**, Theorem 2.1.1]). Since $\dim X \leq n = 2m+1$, we have $\operatorname{def} X \leq 2m+1$. Assume that $\operatorname{def} X < 2m = n-1$. Let us set

$$J_j = (I_1 \times \cdots \times I_{j-1} \times \{0\} \times I_{j+1} \times \cdots \times I_{n+1}) \cap X,$$

and

$$K_j = (I_1 \times \cdots \times I_{j-1} \times \{1\} \times I_{j+1} \times \cdots \times I_{n+1}) \cap X$$

for every j, $1 \leq j \leq n-1$. Then J_j and K_j are disjoint compact subsets of X. Thus, by Lemma 2.1, there is a partition T_j in X between J_j and K_j for every j, $1 \leq j \leq n-1$, such that $\bigcap\{T_j : 1 \leq j \leq n-1\}$ is compact. Since $T_j \cap X_k$ is a partition in X_k between $J_j \cap X_k$ and $K_j \cap X_k$, and X_k is closed in $I^n \times \{1/k\}$, there is a partition T_{jk} in $I^n \times \{1/k\}$ between $J_j \cap X_k$ and $K_j \cap X_k$ such that $T_{jk} \cap X_k \subset T_j \cap X_k$ for each j, $1 \leq j \leq n-1$, and each $k \in \mathbb{N}$. Let S_k be a continuum meeting $I^{n-1} \times \{1/6\} \times \{1/k\}$ and $I^{n-1} \times \{5/6\} \times \{1/k\}$ in $I^n \times \{1/k\}$ with $S_k \subset \bigcap\{T_{jk} : 1 \leq j \leq n-1\} \cap (I^{n-1} \times [1/6, 5/6] \times \{1/k\})$ (see [**6**, Lemma 5.2]). Since S_k is connected, by (8) and (9), we have $S_k \not\subset U_{0k} \cup U_{1k}$. Thus we have $S_k \cap X_k \neq \emptyset$ for every $k \in \mathbb{N}$. Obviously, $S_k \cap X_k \subset \cap\{T_{jk} : 1 \leq j \leq n-1\} \cap X_k \subset \bigcap\{T_j : 1 \leq j \leq n-1\}$ and $\{S_k \cap X_k : k \in \mathbb{N}\}$ is discrete in X. This contradicts the compactness of $\bigcap\{T_j : 1 \leq j \leq n-1\}$. Hence we have $\operatorname{def} X \geq n-1 = 2m$.

5. $1 \leq \operatorname{cmp} X \leq m$**.** Note that $\operatorname{cmp} X \leq 0$ if and only if $\operatorname{def} X \leq 0$ (see [**2**, Main Theorem]). Since $\operatorname{def} X \geq 2m > 0$, we have $\operatorname{cmp} X \geq 1$.

We shall prove that $\operatorname{cmp} X \leq m$. To prove this we only consider points of X_0, because $\bigcup\{X_k : k \in \mathbb{N}\}$ is locally compact and open in X. First we shall show the following

Claim. Let $1 \leq l \leq m$. For every $\{B_1, \ldots, B_l\} \in [\mathcal{B}_i]^l$ and any $(k_1, \ldots, k_l) \in \mathbb{N}^l$ we have $\operatorname{cmp} \cap\{\operatorname{Bd}_X B'_j : 1 \leq j \leq l\} \leq m-l$, where $B'_j = (B_j \times [0, 1/k_j)) \cap X$ for each j, $1 \leq j \leq l$.

Proof of Claim. We proceed by downward induction on l.

Step 1. $l = m$.

Let $Y = \bigcap\{\operatorname{Bd}_X B'_j : 1 \leq j \leq m\}$, $y \in Y$, and U be a neighborhood of y in Y. We show that there is a neighborhood V of y in Y such that $V \subset U$ and $\operatorname{Bd}_Y V$ is compact. We may assume that $y \in X_0$. Then, by (1), (2), (4) and (5), there are $B_{m+1} \in \mathcal{B}_i$ and $k \in \mathbb{N}$ such that $y \in (B_{m+1} \times [0, 1/k)) \cap Y \subset U$. Since $\{B_1, \ldots, B_m, B_{m+1}\} \in [\mathcal{B}_i]^{m+1}$, $\bigcap\{\operatorname{Bd}_{I^n} B_j : 1 \leq j \leq m+1\} = F_{ip}$ for some $p \in \mathbb{N}$. Let $V = (B_{m+1} \times [0, 1/q)) \cap Y$, where $q = \max\{k, p\}$. Then V is a neighborhood of y in Y. Obviously, we have $V \subset U$. By (7), it is easy to see that

$$\operatorname{Bd}_Y V \subset \left(\bigcap\{\operatorname{Bd}_{I^n} B_j : 1 \leq j \leq m+1\} \cap \partial I^n\right) \times \{0, 1/(p+1), 1/(p+2), \ldots\} \subset X.$$

Hence $\operatorname{Bd}_Y V$ is compact; therefore, we have $\operatorname{cmp} Y \leq 0 = m-l$.

Step 2. Let $1 \leq l < m$ and suppose that the Claim is satisfied for $l+1$.

Let $Y = \bigcap\{\mathrm{Bd}_X B'_j : 1 \le j \le l\}$, $y \in Y$, and U be a neighborhood of y in Y. We may assume that $y \in X_0$. Take $B_{l+1} \in \mathcal{B}_i$ and $k \in \mathbb{N}$ such that $y \in B'_{l+1} = (B_{l+1} \times [0, 1/k)) \cap X$ and $B'_{l+1} \cap Y \subset U$. Then we have

$$\mathrm{Bd}_X(B'_{l+1} \cap Y) \subset \bigcap\{\mathrm{Bd}_X B'_j : 1 \le j \le l+1\}.$$

By the induction hypothesis, we have

$$\begin{aligned} \operatorname{cmp} \mathrm{Bd}_Y(B'_{l+1} \cap Y) &\le \operatorname{cmp} \bigcap\{\mathrm{Bd}_Y B'_j : 1 \le j \le l+1\} \\ &\le m - (l+1) = m - l - 1. \end{aligned}$$

Hence we have $\operatorname{cmp} Y \le m - l$.

This completes the proof of the Claim.

By the Claim, in particular, $\operatorname{cmp} \mathrm{Bd}_X((B \times [0, 1/k)) \cap X) \le m - 1$ for every $B \in \mathcal{B}_i$ and every $k \in \mathbb{N}$. Since $\{(B \times [0, 1/k)) \cap X : B \in \mathcal{B}_0 \cup \mathcal{B}_1 \text{ and } k \in \mathbb{N}\}$ is an outer base for X_0 in X, we have $\operatorname{cmp} X \le m$.

ADDED IN PROOF. By using the same techniques in §3, the author constructed a separable metrizable space X for which $\operatorname{cmp} X \ne \operatorname{def} X$ (see [2]).

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