THE GAP BETWEEN \( \text{cmp} \, X \) AND \( \text{def} \, X \) CAN BE ARBITRARILY LARGE

TAKASHI KIMURA

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Dedicated to Professor Yukihiro Kodama on his 60th birthday

Abstract. We give an example of a separable metrizable space \( X \) with
\( \text{def} \, X - \text{cmp} \, X = n \) for every \( n \in \mathbb{N} \).

1. Introduction. In this paper all spaces are separable and metrizable.

The compactness degree, \( \text{cmp} \, X \), of a space \( X \) is defined as follows: a space \( X \) satisfies \( \text{cmp} \, X = -1 \) if \( X \) is compact; if \( n \) is a nonnegative integer, then \( \text{cmp} \, X \leq n \) means that each point of \( X \) has arbitrarily small neighborhoods \( U \) with \( \text{cmp} \, \text{Bd} \, U \leq n - 1 \). We put \( \text{cmp} \, X = n \) if \( \text{cmp} \, X \leq n \) and \( \text{cmp} \, X < n - 1 \). If there is no integer \( n \) for which \( \text{cmp} \, X \leq n \), then we put \( \text{cmp} \, X = \infty \).

The compactness deficiency, \( \text{def} \, X \), of a space \( X \) is the least integer \( n \) for which \( X \) has a compactification \( \alpha \, X \) with \( \dim(\alpha \, X - X) \leq n \). We allow \( n \) to be \( \infty \).

In general, the inequality \( \text{cmp} \, X \leq \text{def} \, X \) holds. The well-known conjecture of J. de Groot (see [2]) that \( \text{cmp} \, X = \text{def} \, X \) has been negatively solved by R. Pol [5]; the space \( X \) of Pol’s example has \( \text{cmp} \, X = 1 \) and \( \text{def} \, X = 2 \). In the review of R. Pol’s paper [5], J. van Mill [3] states “It seems still to be open whether the gap between \( \text{cmp} \, X \) and \( \text{def} \, X \) can be arbitrarily large.”

The purpose of this paper is to answer this question affirmatively. Namely, we shall give the following example.

Example. For every \( n \in \mathbb{N} \) there exists a space \( X \) such that \( \text{def} \, X - \text{cmp} \, X = n \).

2. Preliminaries. Let \( S \) be a collection of subsets of a space \( X \). Then we shall write \([S]^n\) for \( \{T: T \subseteq S \text{ with } |T| = n\} \), \( \text{Bd} \, S \) for \( \{\text{Bd} \, S: S \in S\} \) and \( \bigcap S \) for \( \bigcap\{S: S \in S\} \).

Let \( Y \) be a subspace of a space \( X \) and \( \mathcal{U} \) a collection of open subsets of \( X \). Then \( \mathcal{U} \) is an outer base for \( Y \) in \( X \) if for every \( y \in Y \) and any neighborhood \( V \) of \( y \) in \( X \) there is \( U \in \mathcal{U} \) such that \( y \in U \subset V \).

The following lemma is needed in §4; the proof is straightforward.

2.1. Lemma. Let \( X \) be a space with \( \text{def} \, X < n \) and \( \{(E_j, F_j): 1 \leq j \leq n\} \) a collection of pairs of disjoint compact subsets of \( X \). Then for each \( j, 1 \leq j \leq n \), there is a partition \( T_j \) in \( X \) between \( E_j \) and \( F_j \) such that \( \bigcap\{T_j: 1 \leq j \leq n\} \) is compact.
To show our example it suffices to construct a space $Y$ with $n \leq \text{def } Y - \text{cmp } Y < \infty$. Indeed, for this space $Y$ we construct another space $Z$ with $\text{cmp } Z = \text{def } Z = \text{def } Y - n$; such a space exists (see [2, Theorem 3.1.1]). Let $X = Y \oplus Z$ be the topological sum of $Y$ and $Z$. Then
\[
\text{cmp } X = \max\{\text{cmp } Y, \text{cmp } Z\} = \text{cmp } Z = \text{def } Y - n
\]
and
\[
\text{def } X = \max\{\text{def } Y, \text{def } Z\} = \text{def } Y.
\]
Thus we have $\text{def } X - \text{cmp } X = n$.

In the next section we shall construct a space $X$ such that $m \leq \text{def } X - \text{cmp } X \leq 2m$ for every $m \in \mathbb{N}$.

Throughout the rest of this paper, we shall fix a positive integer $m$ and put $n = 2m + 1$. Let $I = [0,1]$ be the closed unit interval.

3. Construction. Let
\[
\partial I^n = \{(x_j) \in I^n : x_j = 0 \text{ or } 1 \text{ for some } j, \; 1 \leq j \leq n\}
\]
be the combinatorial boundary of the $n$-dimensional cube $I^n$. We take countable, dense subsets $D_0$ and $D_1$ in $(0,1)$ with $D_0 \cap D_1 = \emptyset$. Let us set
\[
M_i = \{(x_j) \in (0,1)^n : |\{j : x_j \in D_i\}| \geq n - m\},
\]
and
\[
L_i = (0,1)^n - M_i
\]
for each $i = 0, 1$. Then, obviously, $M_0 \cap M_1 = \emptyset$ and by [1, 1.8.5], $\dim L_i = m$.

Then, by [4, 12.12–13], there are two collections $\mathcal{B}_0$ and $\mathcal{B}_1$ of open subsets of $I^n$ satisfying the following conditions (1) to (6) below:

1. $\mathcal{B}_0$ is an outer base for $(I^{n-1} \times [0, \frac{2}{3})) \cap \partial I^n$ in $I^n$,
2. $\mathcal{B}_1$ is an outer base for $(I^{n-1} \times (\frac{1}{3}, 1]) \cap \partial I^n$ in $I^n$,
3. $\mathcal{F} \cap L_i = \emptyset$ for every $\mathcal{F} \in [\mathcal{B}_i]^{m+1}$ and each $i = 0, 1$,
4. $\text{Cl } B \subset I^{n-1} \times [0, \frac{2}{3})$ for every $B \in \mathcal{B}_0$,
5. $\text{Cl } B \subset I^{n-1} \times (\frac{1}{3}, 1]$ for every $B \in \mathcal{B}_1$, and
6. $|B_i| = \omega$ for each $i = 0, 1$.

By (6), $[\mathcal{B}_i]^{m+1}$ is countable; therefore, we enumerate it as $[\mathcal{B}_i]^{m+1} = \{\mathcal{F}_j : j \in \mathbb{N}\}$. Let us see $E_{ij} = \bigcap \mathcal{F}_{ij}$, and let
\[
E_{ik} = \bigcup \{F_{ij} : j \leq k\} - \partial I^n
\]
for $i = 0, 1$ and $k \in \mathbb{N}$. Then, by (3), we have $E_{0k} \cap E_{1k} \subset M_0 \cap M_1 = \emptyset$. Thus $E_{0k}$ and $E_{1k}$ are disjoint closed subsets of $(0,1)^n$, therefore we can take disjoint open subsets $U_{0k}$ and $U_{1k}$ in $(0,1)^n$ such that

7. $E_{ik} \subset U_{ik}$ for each $i = 0, 1$,
8. $U_{0k} \subset I^{n-1} \times [0, \frac{2}{3})$, and
9. $U_{1k} \subset I^{n-1} \times (\frac{1}{3}, 1]$.

Let us set
\[
X_k = (I^n - U_{0k} \cup U_{1k}) \times \{1/k\} \text{ for every } k \in \mathbb{N},
\]
$X_0 = \partial I^n \times \{0\}$, and
\[
X = \bigcup \{X_k : k = 0, 1, 2, \ldots\}.
\]
We regard $X$ as the subspace of the $(n + 1)$-dimensional cube
\[ I^{n+1} = \prod \{I_j : 1 \leq j \leq n + 1\}, \]
where $I_j$ is the copy of $I$.

4. $2m \leq \text{def } X \leq 2m + 1$. Note that $\text{def } Y \leq \dim Y$ for every space $Y$ (see [5, Theorem 2.1.1]). Since $\dim X \leq n = 2m + 1$, we have $\text{def } X \leq 2m + 1$. Assume that $\text{def } X < 2m = n - 1$. Let us set
\[ J_j = (I_1 \times \cdots \times I_{j-1} \times \{0\} \times I_{j+1} \times \cdots \times I_{n+1}) \cap X, \]
and
\[ K_j = (I_1 \times \cdots \times I_{j-1} \times \{1\} \times I_{j+1} \times \cdots \times I_{n+1}) \cap X \]
for every $j$, $1 \leq j \leq n - 1$. Then $J_j$ and $K_j$ are disjoint compact subsets of $X$. Thus, by Lemma 2.1, there is a partition $T_j$ in $X$ between $J_j$ and $K_j$ for every $j$, $1 \leq j \leq n - 1$, such that $\bigcap \{T_j : 1 \leq j \leq n - 1\}$ is compact. Since $T_j \cap X_k$ is a partition in $X_k$ between $J_k \cap X_k$ and $K_k \cap X_k$, and $X_k$ is closed in $I^n \times \{1/k\}$, there is a partition $T_{jk}$ in $I^n \times \{1/k\}$ between $T_j \cap X_k$ and $K_j \cap X_k$ such that $T_{jk} \cap X_k \subset T_j \cap X_k$ for each $j$, $1 \leq j \leq n - 1$, and each $k \in \mathbb{N}$. Let $S_k$ be a continuum meeting $I^{n-1} \times \{1/6\} \times \{1/k\}$ and $I^{n-1} \times \{5/6\} \times \{1/k\}$ in $I^n \times \{1/k\}$ with $S_k \subset \bigcap \{T_{jk} : 1 \leq j \leq n - 1\} \cap (I^{n-1} \times [1/6, 5/6] \times \{1/k\})$ (see [6, Lemma 5.2]). Since $S_k$ is connected, by (8) and (9), we have $S_k \subset U_{0k} \cup U_{1k}$. Thus we have $S_k \cap X_k \neq \emptyset$ for every $k \in \mathbb{N}$. Obviously, $S_k \cap X_k \subset \bigcap \{T_j : 1 \leq j \leq n - 1\} \cap X_k \subset \bigcap \{T_j : 1 \leq j \leq n - 1\}$ and $\{S_k \cap X_k : k \in \mathbb{N}\}$ is discrete in $X$. This contradicts the compactness of $\bigcap \{T_j : 1 \leq j \leq n - 1\}$. Hence we have $\text{def } X \geq n - 1 = 2m$. 

5. $1 \leq \text{cmp } X \leq m$. Note that $\text{cmp } X \leq 0$ if and only if $\text{def } X \leq 0$ (see [2, Main Theorem]). Since $\text{def } X \geq 2m > 0$, we have $\text{cmp } X \geq 1$.

We shall prove that $\text{cmp } X \leq m$. To prove this we only consider points of $X_0$, because $\bigcup \{X_k : k \in \mathbb{N}\}$ is locally compact and open in $X$. First we shall show the following

Claim. Let $1 \leq l \leq m$. For every \( \{B_1, \ldots, B_l\} \in [S_l]^l \) and any \((k_1, \ldots, k_l) \in \mathbb{N}^l\) we have $\text{cmp } \cap \{\text{Bd}_X B_j' : 1 \leq j \leq l\} \leq m - l$, where $B_j' = (B_j \times [0, 1/k_j]) \cap X$ for each $j$, $1 \leq j \leq l$.

Proof of Claim. We proceed by downward induction on $l$.

Step 1. $l = m$.

Let \( Y = \bigcap \{\text{Bd}_X B_j' : 1 \leq j \leq m\}, \ y \in Y, \) and $U$ be a neighborhood of $y$ in $Y$. We show that there is a neighborhood $V$ of $y$ in $Y$ such that $V \subset U$ and $\text{Bd}_V V$ is compact. We may assume that $y \in X_0$. Then, by (1), (2), (4) and (5), there are $B_{m+1} \in B_i$ and $k \in \mathbb{N}$ such that $y \in (B_{m+1} \times [0, 1/k]) \cap \bigcap \{B_j : 1 \leq j \leq m + 1\} = F_p$ for some $p \in \mathbb{N}$. Let $V = (B_{m+1} \times [0, 1/q]) \cap Y$, where $q = \max\{k, p\}$. Then $V$ is a neighborhood of $y$ in $Y$. Obviously, we have $V \subset U$. By (7), it is easy to see that
\[ \text{Bd}_V V \subset \left( \bigcap \{\text{Bd}_X B_j' : 1 \leq j \leq m + 1\} \cap \partial I^n \right) \times \{0, 1/(p+1), 1/(p+2), \ldots\} \subset X. \]
Hence $\text{Bd}_V V$ is compact; therefore, we have $\text{cmp } Y \leq 0 = m - l$.

Step 2. Let $1 \leq l < m$ and suppose that the Claim is satisfied for $l - 1$. 

Let \( Y = \bigcap \{ \text{Bd}_X B_j' : 1 \leq j \leq l \} \), \( y \in Y \), and \( U \) be a neighborhood of \( y \) in \( Y \). We may assume that \( y \in X_0 \). Take \( B_{l+1} \in \mathcal{B}_i \) and \( k \in \mathbb{N} \) such that \( y \in B_{l+1}' = (B_{l+1} \times [0, 1/k)) \cap X \) and \( B_{l+1}' \cap Y \subset U \). Then we have
\[
\text{Bd}_X(B_{l+1}' \cap Y) \subset \bigcap \{ \text{Bd}_X B_j' : 1 \leq j \leq l + 1 \}.
\]
By the induction hypothesis, we have
\[
\text{cmp}_B \text{Bd}_Y(B_{l+1}' \cap Y) \leq \text{cmp}_B \bigcap \{ \text{Bd}_Y B_j' : 1 \leq j \leq l + 1 \}
\leq m - (l + 1) = m - l - 1.
\]
Hence we have \( \text{cmp} Y \leq m - l \).

This completes the proof of the Claim.

By the Claim, in particular, \( \text{cmp} \text{Bd}_X((B \times [0, 1/k)) \cap X) \leq m - 1 \) for every \( B \in \mathcal{B}_i \) and every \( k \in \mathbb{N} \). Since \( \{(B \times [0, 1/k)) \cap X : B \in \mathcal{B}_0 \cup \mathcal{B}_1 \text{ and } k \in \mathbb{N}\} \) is an outer base for \( X_0 \) in \( X \), we have \( \text{cmp} X \leq m \).

ADDED IN PROOF. By using the same techniques in §3, the author constructed a separable metrizable space \( X \) for which \( \text{cmp} X \neq \text{def} X \) (see [2]).

REFERENCES


Institute of Mathematics, University of Tsukuba, Sakura-mura, Niihari-gun, Ibaraki, 305, Japan