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SHORTER NOTES

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AN ELEMENTARY SECTION OF A BUNDLE

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ABSTRACT. We use the Cayley algebra and triality to provide an explicit section of a principal G_2-bundle over S^7. This section is the basic ingredient for a direct, elementary, proof that $\pi_6 G_2 \cong \mathbf{Z}_3$, $\pi_6 SU(3) \cong \mathbf{Z}_6$ and $\pi_6 S^3 \cong \mathbf{Z}_{12}$

In this note we use simple algebra to exhibit a section of the pull-back of the bundle

$$G_2 --- \mathrm{Spin}(7) \to S^7 \tag{1}$$

by the map $a \to a^3$ from S^7 to itself. Here G_2 is the automorphism group of the Cayley numbers $\mathbf{K}$. Once it is established that (1) is nontrivial, our section furnishes, also, an elementary proof that $\pi_6(G_2) \cong \mathbf{Z}_3$ (see [**M**]).

Our motivation comes from the account of bundles over S^7 as it appears in [**W**, Appendix A], and in [**P**, Chapter 21]. We rely on these two references for our notation.

First observe that the following Moufang identities, [**H-L**], can be also proved along the lines of 4.21 of [**W**]:

For any a, x, y in $\mathbf{K}$ we have

$$(axa)y = a(x(ay)) \quad \text{and} \quad x(aya) = ((xa)y)a.$$

Let $\|a\| = 1$ now.

These two are equivalent to

$$a(xy) = (axa)(\overline{a}y) \quad \text{and} \quad (xy)\overline{a} = (xa)(\overline{a}y\overline{a}).$$

Triality [**C**, **P**, **W**] applied here, says that to the map A in $SO(8)$ with $A(\theta) = a\theta$ corresponds to $\pm(B, C)$, each in $SO(8)$, with $B(\xi) = a\xi a$ and $C(\eta) = \overline{a}\eta$. Similarly for the map $A_1(\theta) = \theta\overline{a}$. Multiplying these relations we get that for a in S^7 and ξ, η in $\mathbf{K}$,

$$a(\xi\eta)\overline{a} = (a\xi a^2)(\overline{a}^2\eta\overline{a}). \tag{2}$$

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Viewed as a special case of triality, (2) implies that the map $\chi\colon S^7 \to \mathrm{Spin}(8)$, with $\chi(a) := (\sigma_0, \sigma_1, \sigma)$ is well defined, where

$$\sigma_0(\xi) = a\xi a^2, \quad \sigma_1(\eta) = \overline{a}^2\eta\overline{a} \quad \text{and} \quad \sigma(\theta) = a\theta\overline{a}.$$

Since $\sigma(1) = 1$, the image of χ lies in Spin(7) and since $\sigma_0(1) = a^3$ (or equivalently $\sigma_1(1) = \overline{a}^3$) our map provides the desired section.

Another immediate consequence of (2) is that the image τ_a of $\tau\colon S^7 \to SO(7)$, with $\tau_a(x) = ax\overline{a}$, lies in G_2 iff $a^3 = \pm 1$ [**W**].

Observe that $\pi_6 G_2 \cong \mathbf{Z}_3$ together with the homotopy ladder of diagrams 21.6, 21.7 of [**P**] or of the diagram in p. 714 of [**W**] and the following elementary facts:

(i) the inclusions of Spin(5) in Spin(6) and Spin(6) in Spin(7) induce multiplication by 2 on the π_7-level. (All π_7's involved are isomorphic to $\mathbf{Z}$),

(ii) $\pi_6\,\mathrm{Spin}(k) = 0$ for $k \geq 5$,

imply easily that $\pi_6 SU(3) \cong \mathbf{Z}_6$ and $\pi_6 S^3 \cong \mathbf{Z}_{12}$ [**M** and **S**].

ADDED IN PROOF. The author noticed that formula (2) was known: H. Toda, Y. Saito and T. Yokota, *Note on the generator of* $\pi_7 SO(n)$, Mem. Coll. Sci. Univ. Kyoto Ser. A **30** (1957), 227–230.

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