THE SCHUR SUBGROUP OF THE BRAUER GROUP
OF CYCLOTONIC RINGS OF INTEGERS

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(Communicated by Bhama Srinivasan)

ABSTRACT. Let $K$ be a finite abelian extension of the rational numbers $\mathbb{Q}$. Let $S$ be a finite set of primes of $K$ including the infinite ones, and let $\mathfrak{o}$ be the ring of $S$-integers in $K$. Then the Schur subgroup $S(\mathfrak{o})$ of the Brauer group $B(\mathfrak{o})$ is defined, in analogy with $S(K)$, via representations of finite groups on finitely generated projective $\mathfrak{o}$-modules. It is easy to see that $S(\mathfrak{o}) \subseteq S(K) \cap B(\mathfrak{o})$. We shall show that there is equality in the case of $K$ a purely cyclotomic extension $\mathbb{Q}(\varepsilon_m)$ of $\mathbb{Q}$ (where $\varepsilon_m$ is an $m$th root of $1$).

Introduction. For any commutative ring $\mathfrak{o}$, one can define the Schur subgroup $S(\mathfrak{o})$ of the Brauer group $B(\mathfrak{o})$ as follows: Let $G$ be a finite group, and $\rho: G \to \text{GL}(M)$ a homomorphism of $G$ to the automorphism group of a projective module $M$ over $\mathfrak{o}$ with the property that the $\mathfrak{o}$-span $\rho G$ of $\rho G$ is an Azumaya algebra over $\mathfrak{o}$. In this case we denote the Brauer class $[\rho G]$ by $\beta(\rho)$. Then $S(\mathfrak{o})$ consists of the Brauer classes $\beta(\rho)$ as $\rho$ runs over all such homomorphisms of all finite groups $G$. Alternatively, one can define $S(\mathfrak{o})$ to be the set of all Brauer classes $[\Lambda]$ where $\Lambda$ runs over all Azumaya algebras which are homomorphic images of some group algebra $\mathfrak{o}G$ ($G$ finite). It is easy to see, as in the case $\mathfrak{o} = \text{a field}$, that $S(\mathfrak{o})$ is indeed a subgroup of $B(\mathfrak{o})$ (see [D-M 1]).

Suppose now that the canonical map $B(\mathfrak{o}) \to B(K)$ is injective (this is the case if $\mathfrak{o}$ is a Dedekind domain, e.g., see [O-S]). Then it is clear that

$$S(\mathfrak{o}) \subseteq S(K) \cap B(\mathfrak{o})$$

and E. B. Williams has asked for conditions under which there is equality in the following situation: $K$ is a finite abelian extension of $\mathbb{Q}$, and $\mathfrak{o}$ is the ring of $S$-integers in $K$ for a finite set of places $S$ of $K$. We shall prove the following theorem.

**THEOREM.** Let $K$ be a cyclotomic field $\mathbb{Q}(\varepsilon_m)$ where $m$ is a positive integer, and let $S$ be any finite set of primes (containing the infinite primes). Then if $\mathfrak{o}$ is the ring of $S$-integers,

$$S(\mathfrak{o}) = S(K) \cap B(\mathfrak{o}).$$

For other results on $S(\mathfrak{o})$, see [D-M 1 and D-M 2].
**The proof.** For any abelian group $A$ and any positive integer $q$, let $A_q$ denote the $q$-primary subgroup of $A$—that is, the subgroup of $A$ of those elements whose order divides some power of $q$.

Let $K = \mathbb{Q}(\varepsilon_m)$. We may assume that $m \neq 2 \pmod{4}$. We must show that an arbitrary element $a \in S(K) \cap B(\mathcal{O})$ is in $S(\mathcal{O})$. Since $B(K)$ is a torsion group, we may write $a = \sum \alpha_q$ where $q$ runs over the rational primes and $\alpha_q$ is the $q$-primary component of $a$. This allows us to assume that $\alpha = \alpha_q$, i.e. that $\alpha$ is $q$-primary. By the “root of unity theorem” of Benard (see Theorem 6.1, [Y]), we can also assume that $q|2m$.

If $p$ is any rational prime, possibly $\infty$, and $L$ is an algebraic number field, then the $p$-local invariants of a Brauer class $\gamma \in B(L)$ are, by definition, the Hasse invariants (with values in $\mathbb{Q}/\mathbb{Z}$) of $\gamma$ at the primes of $L$ which lie above $p$—see Definition 6.4, [Y]. The Benard-Schacher theorem says that the Hasse invariants of each $\gamma$ in $S(K)$ are “uniformly distributed”, i.e. for each $p$, the $p$-local invariants of $\gamma$ are determined (in a very explicit way) by any one of them (Theorem 6.1, [Y]). The $p$-local component of $\gamma$ is defined to be the Brauer class $\gamma(p)$ whose $p$-invariants coincide with those of $\gamma$ and whose other Hasse invariants are $0$. Thus $\gamma = \sum_p \gamma(p)$.

Let $M/L$ be a Galois extension with Galois group $\mathcal{G}(M/L)$. We shall denote the cohomology group $H^2(\mathcal{G}(M/L), M^*)$ by $H^2(M/L)$ and identify it with a subgroup of $B(L)$ in the usual way. Let $\mu(M)$ denote the group of roots of unity in $M$. There is a canonical map of $H^2(\mathcal{G}(M/L), \mu(M))$ into $H^2(\mathcal{G}(M/L), M^*)$ whose image will be denoted by $H^2_\mathcal{O}(M/L)$. The Brauer-Witt theorem (Chapter 3, [Y]) says that $S(L)$ is the union of all $H^2_\mathcal{O}(M/L)$ as $M$ runs over all cyclotomic extensions $M = L(\varepsilon_n)$ of $L$.

Suppose that $q | m$.

**Lemma 1.** The $p$-components $\alpha(p)$ of $\alpha$ are also in $S(K) \cap B(\mathcal{O})$. Moreover $\alpha(p) = K \otimes \beta$ where $\beta \in H^2_\mathcal{O}(\mathbb{Q}(\varepsilon_{qr}))/\mathbb{Q}(\varepsilon_{qr}))$ is a $p$-local $(\beta = \beta(p))$ Brauer class of $\mathbb{Q}(\varepsilon_{qr})$.

Here $K \otimes \beta$ stands for the Brauer class of $K$ obtained from $\beta$ by “restriction” (extension of scalars).

**Proof.** By a theorem of Janusz [J], there is a $\beta \in S(\mathbb{Q}(\varepsilon_{qr}))$ such that $\alpha = K \otimes \beta$. Benard, Schacher, and Yamada have characterized $S(\mathbb{Q}(\varepsilon_{qr}))$ in terms of Hasse invariants (pp. 135–139, [Y]). Namely $S(\mathbb{Q}(\varepsilon_{qr}))$ consists of all uniformly distributed Brauer classes in $B(\mathbb{Q}(\varepsilon_{qr}))$ whose $p$-local invariants have order dividing $(p - 1, q^r)$ for each (rational) prime $p$; there is one exception: if $q = 2$ and $p \equiv -1 \pmod{2^r}$, then the $p$-local invariant is $0$. It follows that $S(\mathbb{Q}(\varepsilon_{qr}))$ is the direct sum of its $p$-local components—the latter are the (cyclic) subgroups of $S(\mathbb{Q}(\varepsilon_{qr}))$ which are $0$ locally at all primes of $\mathbb{Q}(\varepsilon_{qr})$ not above $p$. Therefore if we express $\beta$ as the sum $\sum \beta(p)$ of its $p$-local components, each component lies in $S(\mathbb{Q}(\varepsilon_{qr}))$ and furthermore we can assume it is $0$ if the corresponding $p$-component $\alpha(p)$ of $\alpha$ is $0$. It is also clear that $\alpha(p) = K \otimes \beta(p)$ and, in particular that $\alpha(p) \in S(K)$. Since $B(\mathcal{O})$ consists of the classes in $B(K)$ whose Hasse invariants are $0$ outside of $\mathcal{S}$ (cf. Theorem 6.33 and Proposition 6.34, [O-S]), it follows at once that $\alpha(p) \in S(K) \cap B(\mathcal{O})$. It follows from Lemma 8.5 and Theorem 8.6, [Y], that $\beta(p) \in H^2_\mathcal{O}(\mathbb{Q}(\varepsilon_{pq^n}))/\mathbb{Q}(\varepsilon_{qr}))$. This finishes the proof of Lemma 1.

By Lemma 1, we can now assume that $\alpha$ is $p$-local as well being $q$-primary.
LEMMA 2. The cohomology class $\alpha$ contains a cocycle $f \in Z^2(K(\varepsilon_p)/K)$ all of whose values are in $\langle \varepsilon_{q^r} \rangle$.

PROOF. Because of our identifications,
$$H^2_c(Q(\varepsilon_{pq^r})/Q(\varepsilon_{q^r})) \subseteq H^2_c(Q(\varepsilon_{pm})/Q(\varepsilon_{q^r})).$$
Therefore since $\alpha = K \otimes \beta$ where $\beta$ is as in Lemma 1, we get $\alpha \in H^2_c(K(\varepsilon_p)/K)$ and so Lemma 2 follows at once since $\alpha$ is $q$-primary.

LEMMA 3. If $\alpha \neq 0$, then $p$ lies below a prime of $S$ and does not divide $m$.

PROOF. If $p$ were a divisor of $m$, then $K$ would contain $\varepsilon_p$ resp. $\varepsilon_4$ if $p = 2$; by Proposition 4.8 and Corollary 5.4 of [Y], the Schur subgroup of a completion $K_p$ of $K$, at any prime $p$ lying over $p$, would then be trivial, so $\alpha$ also would be trivial. On the other hand, the fact that $\alpha \in B(\mathfrak{o})$ means exactly that its local components are 0 at all $p$ not in $S$. \( \square \)

We now define $A$ be the the crossed-product algebra
$$A = (K(\varepsilon_p)/K, f) = \sum_{\sigma \in \mathcal{F}} K(\varepsilon_p)u_{\sigma}$$
where $\mathcal{F} = \mathcal{F}(K(\varepsilon_p)/K)$. Of course $A \in \alpha$. Similarly we define the order
$$\Lambda = (\mathfrak{o}[\varepsilon_p]/\mathfrak{o}, f) = \sum_{\sigma \in \mathcal{F}} \mathfrak{o}[\varepsilon_p]u_{\sigma}.$$ 
Thus $A = K \otimes \Lambda$.

LEMMA 4. $\Lambda$ is an Azumaya algebra over $\mathfrak{o}$ with Brauer class $[\Lambda] = \alpha$.

PROOF. Let $p$ be any prime not in $S$. Then
$$\hat{\delta}_p \otimes \Lambda = \sum_{\sigma \in \mathcal{F}} \hat{\delta}_p \otimes \mathfrak{o}[\varepsilon_p]u_{\sigma}$$
is an $\hat{\delta}_p$-order in the $K_p$-algebra
$$K_p \otimes A = \sum_{\sigma \in \mathcal{F}} K_p \otimes K(\varepsilon_p)u_{\sigma}.$$ 
Let $L_1, \ldots, L_g$ be the completions of $L = K(\varepsilon_p)$ at the primes lying above $p$. Then the $K_p$-algebra $K_p \otimes K(\varepsilon_p)$ is (isomorphic to) the direct sum $L_1 \oplus \cdots \oplus L_g$. We shall now show that $\hat{\delta}_p \otimes o[\varepsilon_p]$ is likewise the direct sum of the rings of integers $\mathfrak{D}_i$ of the $L_i$, after identifying by means of this isomorphism.

Since $\hat{\delta}_p \otimes o[\varepsilon_p] \subseteq \bigoplus \mathfrak{D}_i$ and both are $\hat{\delta}_p$-orders on a separable algebra, it suffices to show that they have the same discriminant (cf. [MO]). Now the discriminant of $\hat{\delta}_p \otimes o[\varepsilon_p]$ is $d(o[\varepsilon_p]/o)\hat{\delta}_p$ where $d(o[\varepsilon_p]/o)$ is the discriminant of $o[\varepsilon_p]/o$. The discriminant of $\bigoplus \mathfrak{D}_i$ is the product of the $d(\mathfrak{D}_i/\hat{\delta}_p)$ and thus is equal to $d(o[\varepsilon_p]/o)\hat{\delta}_p$ by Proposition 5, Chapter I, [C-F].

Let $\mathcal{F}_1 \subseteq \mathcal{F}$ be the stabilizer of the summand $L_1$—it is the decomposition group of the prime belonging to $L_1$. Consider the crossed-product order $\Lambda_1 = (\mathcal{D}_1/\hat{\delta}_p, f_1)$ where $f_1$ is the restriction of $f$ to $\mathcal{F}_1$. Now $f_1$ is split since $f$ splits at $p$. Since $p$ does not ramify in $\mathcal{D}_1$, $\Lambda_1$ is a maximal order by a theorem of Auslander and Goldman—see Theorem 28.5, [C-RI]. Since $\hat{\delta}_p \otimes o[\varepsilon_p]$ is the direct sum of the $\mathfrak{D}_i$, it
follows that $\Lambda$ is also maximal by a theorem of Merklen (Proposition 1, (iv), [M]). Since $K \otimes \Lambda = A$ and the local Hasse invariants of $A$ are all 0 at the primes of $\mathfrak{o}$ (i.e. the primes $p \not\in S$), it follows that $\Lambda$ is an Azumaya algebra over $\mathfrak{o}$ and $\Lambda \in \alpha$ (cf. [D-I]). □

Thus to prove our theorem, it suffices to prove that $\Lambda \in S(\mathfrak{o})$.

Let $G$ be the extension of $S$ by $\mu(L)$, corresponding to the cocycle $f$. Then we can view $G$ as the group generated by $\mu(L)$ and the $u_{\sigma}$ in $\Lambda$. Moreover it is clear that $G$ spans $\Lambda$ over $\mathfrak{o}$, and so $\Lambda \in S(\mathfrak{o})$ as desired. □

REFERENCES


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