THE MIDDLE ANNIHILATOR CONJECTURE
FOR EMBEDDABLE RINGS

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ABSTRACT. It is shown that any ring which can be embedded in an Artinian
ring has just finitely many middle annihilator primes. In particular, this proves
the middle annihilator conjecture for a large class of Noetherian rings.

In recent years, key questions for Noetherian ring theory have dealt with the
behavior of various types of prime ideals. The minimal primes, prime annihilators
and affiliated primes are both useful and interesting in their own right. These ideals
are all examples of middle annihilator prime ideals, and so the latter class has come
under examination.

In particular, the question arises of to what extent various results for the classes
of prime ideals mentioned above have analogs for middle annihilator prime
ideals. The most immediate question of this sort asks whether a Noetherian ring
has just finitely many middle annihilator primes. The result of this note answers
this question in the affirmative for a large class of (not necessarily Noetherian)
rings.

THEOREM. Let $R$ be a ring which can be embedded into an Artinian ring. Then
$R$ has just finitely many middle annihilator prime ideals.

We begin by placing this result in context. Middle annihilator ideals were first
defined in an unpublished note of Kaplansky.

DEFINITION. Let $\mathbb{R}$ be a ring with ideals $A$ and $B$ such that $AB \neq (0)$. Let
$I = \{x \in \mathbb{R} | AxB = 0\}$. Then $I$ is the middle annihilator of $A$ and $B$; write
$I = \text{Mid}(A, B)$.

Note that every right annihilator ideal is a middle annihilator, and that maximal
middle annihilators are prime.

Kaplansky notes that in a ring with the ascending chain condition on ideals,
every minimal prime is a middle annihilator. Conversely, Krause [8] shows that a
Noetherian ring admits a classical Artinian ring of quotients if and only if all of its
middle annihilator primes are minimal.

When such a quotient ring exists, then the middle annihilator prime ideals are
manifestly finite in number. Krause also established this result for fully bounded
Noetherian rings; this has been obtained independently by Small and Stafford [11].
(The latter authors also show by example that middle annihilator primes need not
be affiliated.) Goldie and Krause [4] prove that nonsingular Noethrian rings have
just finitely many middle annihilator prime ideals.

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A unifying theme of the results cited here is that the rings in question admit embeddings into Artinian rings. (For FBN rings, the embedding was established in [6]; while for nonsingular Noether rings it can be found in [2, p. 97].) Thus, the theorem given here subsumes the results mentioned above. Recently, Blair and Small [1] have shown that Krull homogenous Noetherian rings are embeddable. In addition, factors of group rings of polycyclic-by-finite groups are known to be embeddable [7], and many Noetherian rings admit ad hoc embeddings. However, the result given here is not comprehensive: While [3] shows that \( U(sl_2(\mathbb{C})) \) possesses a nonembeddable factor, in [11] it is proved that factors of enveloping algebras of finite-dimensional Lie algebras have just finitely many middle annihilator prime ideals.

**Theorem.** Let \( R \) be a ring which can be embedded into an Artinian ring \( S \). Then \( R \) has just finitely many middle annihilator prime ideals. The number of such ideals is less than or equal to the length of \( sS \).

**Proof.** Let \( \{P_i\} \) be the set of prime middle annihilators of \( R \); say \( P_i = \text{Mid}(A_i, B_i) \). Then \( P_i \) is the right annihilator of \( (A_i/A'_i) \), where \( A'_i \) is the left annihilator of \( B_i \). Without loss of generality, we may assume that \( A_i \) is the left annihilator of \( P_i \). Write \( P_i = \text{r}(A_i/A'_i) \).

Considered as an \( S-R \) bimodule, \( S \) certainly has finite length. Note that for any \( i \), \( SA_i \cap R = A_i \), \( SA'_i \cap R = A'_i \), and that \( P_i = \text{r}(SA_i/SA'_i) \). Thus we may pick an \( S-R \) bimodule composition series for \( S \) which passes through \( SA_i \) and \( SA'_i \), say,

\[
\begin{align*}
(0) &= C_0 \subset C_1 \subset \cdots \subset C_n = SA'_i \subset \cdots \subset C_{n+m} = SA_i \cdots C_t = S.
\end{align*}
\]

For each \( j \in \{1, 2, \ldots, t\} \), the simplicity of \( C_j/C_{j-1} \) as a bimodule implies that \( \text{r}(C_j/C_{j-1}) \) is a prime ideal of \( R \); call it \( Q_j \). Now, \( C_{n+m}Q_{n+m}Q_{n+m-1} \cdots Q_{n+1} \subset C_n \); thus \( Q_{n+m}Q_{n+m-1} \cdots Q_{n+1} \subset P_i \), and \( Q_j \subset P_i \) for some \( j \in \{n+1, n+2, \ldots, n+m\} \). The reverse inclusion being obvious, we have realized \( P_i \) as the annihilator of a composition factor. Invoking the Jordan-Hölder Theorem, which states that the composition factors from any two series can be paired isomorphically, we see that the set \( \{P_i\} \) is finite. Thus \( R \) has just finitely many middle annihilator prime ideals. The bound on the number of these follows immediately.

The proof of this theorem is reminiscent of that of [12, Theorem 1]. In fact, using this reference we obtain further information about the middle annihilator primes of embeddable \( k \)-algebras (\( k \) a field) as follows. A result of Schofield [9, Theorem 7.13] asserts that every embeddable \( k \)-algebra can be embedded in a simple Artinian ring. Suppose that \( R \) is an embeddable \( k \)-algebra whose prime factors are right Goldie. Let \( S \) be a simple Artinian overring for \( R \). Then, in the terminology of [5], the middle annihilator primes of \( R \) are a subset of the (finite) set of primes affiliated to \( (0) \) in \( S \); in case \( S \) is simple Artinian, the primes affiliated to \( (0) \) are just the annihilators of the composition factors of the series (\( \ast \)).

Let \( X = \{ \text{prime ideals } P \text{ of } R \mid P \text{ is affiliated to } (0) \text{ in } S \} \). Then by [12, Theorem 1], we have

\[
\text{rank } S = \sum_{P \in X} n(P) \text{rank}(R/P)
\]

where each \( n(P) \) is a positive integer.
However, it is not the case that the additivity formula involves only the middle annihilator primes. The following example from [10] illustrates this point.

EXAMPLE. Let $k$ be a field, $A = k[X_1, X_2]$ and define $\rho : A \to A$ by $\rho(X_1) = 0$, $\rho(X_2) = X_1$. Let

$$R = \left\{ \begin{pmatrix} a & b & 0 \\ X_1 c & d & 0 \\ 0 & 0 & \rho(d) \end{pmatrix} \mid a, b, c, d \in A \right\}.$$ 

Then $R$ has rank 2; $R$ can be embedded in the simple Artinian ring of $3 \times 3$ matrices over $k(X_1, X_2)$. Call this ring $S$. Now, $S$ has rank 3; however, the only middle annihilator prime of $R$ is $(0)$. Thus, the additivity formula cannot be based on middle annihilator primes alone.

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REFERENCES


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