ABSTRACT. We define matrix representations of Artin groups over a 2-variable Laurent-polynomial ring and show that in the rank 2 case, the representations are faithful. In the special case of Artin’s braid group, our representation is a version of the Burau representation and our faithfulness theorem is a generalization of the well-known fact that the Burau representation of $B_3$ is faithful.

In [4], Brieskorn and Saito coined the phrase “Artin groups” to denote a certain class of groups, defined by generators and relations, which stand in relationship to arbitrary Coxeter groups much as Artin’s braid group $B_n$ [1] stands in relationship to the symmetric group $S_n$. One of the nice features of Coxeter groups is that they have “standard” representations [6] as groups of matrices over the real numbers preserving a suitably defined bilinear form and that, moreover, these representations are faithful (see [3]). Our purpose here is to show the existence of analogous matrix representations of Artin groups over Laurent-polynomial rings preserving similarly defined sequilinear forms. Unfortunately, except in the simplest cases, the question of faithfulness of these Artin group representations remains open.

In §1, we define Artin groups $G_M$ (by representation), a Hermitian form $J$, and unitary reflections for each given generator of $G_M$; these are defined using a given Coxeter matrix $M$. In §2, we show that the reflections associated to generators of $G_M$ define a matrix representation of $G_M$ (Theorem 1) and that when the presentation of $G_M$ involves 2 generators, this representation is faithful (Theorem 2). We note that in the special case of the braid groups our representation is a version of the Burau representation ([5] or see [2]). The results below are first, a generalization to arbitrary Artin groups of the author’s observation [10] that the Burau representation of $B_n$ is unitary and second, a generalization to arbitrary rank 2 Artin groups of the well-known fact (see [9 or 2]) that the Burau representation of $B_3$ is faithful.

1. Definitions. Let $n$ be a positive integer. A (rank $n$) Coxeter matrix $M$ will be an $n \times n$ symmetric matrix $M = [m(i,j)]$ each of whose entries $m(i,j)$ is a positive integer or $\infty$ such that $m(i,j) = 1$ if and only if $i = j$. Out of a Coxeter matrix $M$, we shall build some presentations and some forms.

To define the presentations, let $X = \{x_1, \ldots, x_n\}$ be a finite set. For $m$ a positive integer, define the symbol $\langle xy \rangle^m$ by the formula

$$\langle xy \rangle^m = \begin{cases} (xy)^k & \text{if } m = 2k, \\ (xy)^k x & \text{if } m = 2k + 1. \end{cases}$$

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Let $M$ be an $n \times n$ Coxeter matrix. $G_M$ will denote the abstract group defined by generators $X = \{x_1, \ldots, x_n\}$ and relations all \( x_i x_j x_i = x_j x_i x_j \) for $1 \leq i < j \leq n$. Throughout, the case $m(i, j) = \infty$ will stand for “no relation”. $G_M$ is the Artin group determined by $M$. $W_M$ will denote $G_M$ modulo the addition relations all $x_i^2 = 1$. Note that in the presence of the relations $x_i^2 = 1$, the defining relations of $G_M$ take the form $(x_i x_j)^{m(i, j)} = 1$. $W_M$ is called the Coxeter group determined by $M$. For the basic properties of Coxeter groups, see [3 or 6]. For a study of Artin groups and their relationship to Coxeter groups, see [4].

We define a symmetric bilinear form $J_1$ associated to $W_M$ and a Hermitian form $J$ associated to $G_M$. To motivate the definitions of $J$, we begin by recalling the (well-known—see [3]) definition of $J_1 = J_1(M)$: $J_1$ is the $n \times n$ matrix $[c_{ij}]$ where $c_{ij} = -2 \cos(\pi/m(i, j))$. Here, we adopt the convention that $\pi/\infty = 0$ so that if $m(i, j) = \infty$ then $c_{ij} = -2$. Note that each $c_{ii} = 2$. Let $V$ denote an $n$-dimensional vector space over $\mathbb{R}$ with basis $\{e_1, \ldots, e_n\}$. Identify each $v \in V$ with the column vector consisting of the coordinates of $v$ with respect to the basis $\{e_1, \ldots, e_n\}$ of $V$. With this convention, if $v \in V$, let $v'$ denote the transpose of $v$ and, for $u, v \in V$, define $\langle u, v \rangle_1 = u' J_1 v$. Thus, $J_1$ defines a symmetric bilinear form on $V$. We use $J_1$ to define a matrix representation $\rho_1$ of $W_M$ on $V$: if $v \in V$ and $x_i \in X$ define $(\rho_1(x_i))(v) = v - \langle e_i, v \rangle_1 e_i$.

It is well known (again see [3]) that $\rho_1$ is a faithful linear representation of $W_M$.

To define $J$, let $\Lambda$ denote the Laurent-polynomial ring $\mathbb{R}[s, s^{-1}, t, t^{-1}]$, where $s$ and $t$ are indeterminates over $\mathbb{R}$. Define $J = J(M)$ to be the $n \times n$ matrix $[a_{ij}]$ over $\Lambda$, where

$$
a_{ij} = \begin{cases} 
-2s \cos(\pi/m(i, j)), & i < j, \\
1 + st, & i = j, \\
-2t \cos(\pi/m(i, j)), & i > j.
\end{cases}
$$

Note that $J_1$ may be obtained from $J$ by substituting $s = t = 1$.

To define analogues of the representation $\rho_1$ of $W_M$ defined above, we introduce an analogue of complex conjugation in the Laurent-polynomial ring $\Lambda$: if $x \in \mathbb{R}$ then, as usual, $\overline{x} = x$; also, $\overline{s} = s^{-1}$ and $\overline{t} = t^{-1}$, extended to $\Lambda$ additively and multiplicatively. Note that if complex numbers of norm 1 are substituted for $s$ and $t$ then we recover ordinary complex conjugation.

We extend the definition of conjugation to matrices entrywise and, if $A$ is a matrix over $\Lambda$, we define $A^* = \overline{A'}$. For example, note that $J^* = s^{-1} t^{-1} J$.

Let $V$ denote a free $\Lambda$-module with basis $\{e_1, \ldots, e_n\}$ and, as above, identify each $v \in V$ with its column vector of coordinates. If $u, v \in V$ define $\langle u, v \rangle = u^* J v$. Finally, we define $\rho$: if $v \in V$ and $x_i \in X$ define

$$(\rho(x_i))(v) = v - \langle e_i, v \rangle e_i.$$ 

We shall see below that $\rho$ provides a matrix representation of the Artin group $G_M$.

Note that $\langle \rho(x_i)(v), s^{-1} t^{-1} (e_i, v) e_i \rangle = v$. It follows that each $\rho(x_i)$ acts invertibly on $V$. In fact, each $\rho(x_i)$ is a pseudo-reflection in the sense of [3]. Also, for each $x_i \in X$ and each $u, v \in V$, we have

$$\langle \rho(x_i)(u), \rho(x_i)(v) \rangle = \langle u, v \rangle.$$ 

Combining this observation with Theorem 1 below, we conclude that $\rho$ is a representation of $G_M$ in a group of unitary matrices.
2. Theorems. In this section, we show that the function $\rho$ defined (on generators) above extends to a representation of the Artin groups $G_M$ and that when $n = 2$, this representation is faithful. (The second result includes the fact that the Burau representation of $B_3$ is faithful—see [9 or 2].)

To prove that $\rho$ defines a representation of $G_M$, we need to show that $\rho$ respects the defining relations of $G_M$. An important observation is the following

**Lemma.** $\det J \neq 0$.

**Proof.** In $\det J$, the coefficients of $(st)^n$ is 1, so $\det J \neq 0$. □

In particular, $(-, -)$ is nondegenerate: if $u \in V$ satisfies $\langle u, v \rangle = 0$ for all $v \in V$, then $u = 0$.

At this point, it is convenient to introduce the field-of-quotients $F$ of $\Lambda$. $F$ is a rational function field over $R$. Extend the definition of conjugation to $F$. Letting $V_F$ denote the $F$-vector space $V \otimes_A F$, extend $(-, -)$ to $V_F$ and also view $\rho$ as a linear transformation on $V_F$. Note that since $(-, -)$ is nondegenerate, if $u \in V_F$ satisfies $u \neq 0$, then $u^\perp = \{v \in V_F | \langle u, v \rangle = 0\}$ is an $(n-1)$-dimensional subspace of $V_F$. Also note that $\rho(e_i)$ is the identity on $e_i^\perp$. Given $i, j$ satisfying $1 \leq i < j \leq n$, let $V_{ij}$ denote the subspace of $V_F$ spanned by $e_i$ and $e_j$, and let $V_{ij}^\perp = e_i^\perp \cap e_j^\perp$. We need the following

**Lemma.** $V_{ij} \cap V_{ij}^\perp = \{0\}$.

**Proof.** Let $v = v_i e_i + v_j e_j \in V_{ij}$ where $v_i, v_j \in \Lambda$. If $v \in V_{ij}^\perp$, then $\langle e_i, v \rangle = \langle e_j, v \rangle = 0$ which leads to the following system of linear equations:

\[
\begin{align*}
sv_i (1 + st) - 2v_j s \cos(\pi/m) &= 0, \\
-2v_i t \cos(\pi/m) + v_j (1 + st) &= 0,
\end{align*}
\]

where $m$ denotes $m(i, j)$. Since the determinant of the coefficient matrix is $\neq 0$ in $\Lambda$, the only solution is $v_i = v_j = 0$, so $v = 0$, as required. □

Noting that the defining relations of $G_M$ each involve exactly two generators, in order to show that $\rho$ respects the defining relations of $G_M$, it suffices to show that each $\langle x_i x_j \rangle^{m(i, j)} = \langle x_j x_i \rangle^{m(i, j)}$ holds under $\rho$ on the subspace $V_{ij}$ of $V_F$.

Let $a$ denote the matrix of $x_i$ and $b$ the matrix of $x_j$ with respect to the basis $e_i, e_j$ of $V_{ij}$. Writing $m$ for $m(i, j)$, it follows that

\[
a = \begin{pmatrix}
-st & 2s \cos(\pi/m) \\
0 & 1
\end{pmatrix}, \quad b = \begin{pmatrix}
1 & 0 \\
2t \cos(\pi/m) & -st
\end{pmatrix}.
\]

Thus it suffices to prove

**Lemma.** The matrices $a$ and $b$ above satisfy $(ab)^m = (ba)^m$.

**Proof.** Adjoin a square root $q$ of $st^{-1}$ to $F$ and let

\[
R = \begin{pmatrix}
0 & q \\
-q^{-1} & 0
\end{pmatrix}.
\]

It is easy to check that $R^2 = I$ and $b = RaR$. It follows that $(ab)^m = (ba)^m$ if and only if $(aR)^m = (Ra)^m$. Clearly, $s^{-1}q(aR)$ and $s^{-1}q(Ra)$ have determinant 1 and trace $2 \cos(\pi/m)$. It follows that $(s^{-1}q(aR))^m = (s^{-1}q(Ra))^m = -I$, as required. □

Thus we have the following theorem.
THEOREM 1. The function \( \rho \) extends to a representation of \( G_M \) in \( GL_n(\Lambda) \).

PROOF. Each relation \( (x_i x_j)^m(i,j) = (x_j x_i)^m(i,j) \) holds under \( \rho \) on \( V_{ij} \) by the lemma and therefore on all of \( V_F \) since \( x_i \) and \( x_j \) are each the identity on \( V_{ij}^\perp \). \( \square \)

Except in the two-generator case, we do not know if the representation \( \rho \) is faithful. Here is the proof in the two-generator case. Let \( A \) and \( B \), respectively, denote the matrices obtained by substituting \( s = 1 \) and \( t = -1 \) in \( a \) and \( b \) above.

LEMMA. The matrix group generated by \( A \) and \( B \) has presentation \( (AB)^m = (BA)^m \) and

\[
(AB)^m = 1 \quad (m \text{ even}),
\]
\[
(AB)^{2m} = 1 \quad (m \text{ odd}).
\]

PROOF. View \( A \) and \( B \) as linear fractional transformations acting on the upper half-plane. Using the fact that the matrix \( AB \) has determinant 1 and trace \( 2 \cos(\pi(1 - (2/m))) \), it follows that \( AB \) satisfies \( (AB)^m = (-1)^m I \). Thus, it suffices to prove that the group of linear fractional transformations generated by \( A \) and \( B \) has defining relations \( (AB)^m = (BA)^m \) and \( (AB)^m = 1 \).

We prove this last fact by exhibiting the group generated by \( A \) and \( B \) as a subgroup of finite index in a suitable triangle group. Let \( R_1, R_2 \) and \( R_3 \) be transformations of the upper half-plane defined by

\[
R_1 = \text{reflection in the imaginary axis } x = 0,
\]
\[
R_2 = \text{reflection in the axis } x = \cos(\pi/m),
\]
\[
R_3 = \text{reflection in the unit circle}.
\]

Then \( R_1, R_2 \) and \( R_3 \) generate a \((2, m, \infty)\) triangle group with presentation (see [7]):

\[
R_1^2 = R_2^2 = R_3^2 = (R_1 R_3)^2 = (R_2 R_3)^m = 1.
\]

Noting that \( R_1(z) = -\bar{z}, R_2(z) = -\bar{z} + 2 \cos(\pi/m) \) and \( R_3(z) = 1/\bar{z} \), it follows that, as linear fractional transformations, \( A = R_2 R_1 \) and \( B = R_3 R_1 R_2 R_3 \). It can be checked that the subgroup of the triangle group generated by \( A \) and \( B \) is normal and has index 2 when \( m \) is odd and index 4 when \( m \) is even. A routine application of the Reidemeister-Schreier algorithm produces the required presentation of the group generated by \( A \) and \( B \). \( \square \)

THEOREM 2. The group of matrices generated by \( a \) and \( b \) has presentation \( \langle ab \rangle^m = \langle ba \rangle^m \).

PROOF. By the Lemma, the substitution produces a group with a presentation consisting of the desired relation together with a further relation \( c = 1 \) where \( c = (ab)^m \) when \( m \) is even and \( c = (ab)^{2m} \) when \( m \) is odd. In either case, \( c \) is a central element in the group defined by \( \langle ab \rangle^m = \langle ba \rangle^m \). It follows that any additional relation between \( a \) and \( b \) must be a nonzero power of \( c \). But any nonzero power of \( c \) has determinant a nonzero power of \( -st \) and is therefore not the identity. Thus the matrix group generated by \( a \) and \( b \) has presentation \( \langle ab \rangle^m = \langle ba \rangle^m \), as desired. \( \square \)
3. **Remarks.** The (reduced) Burau representation of $B_n$ (see [2]) may be obtained by substituting $s = 1$ in the representation $\rho$ of $B_n$ that arises above. In fact, the representation $\rho$ itself is equivalent to the Burau representation: it is possible to conjugate the image of $\rho$ by a diagonal matrix that, in each $\rho(x_i)$, "moves the t's above the diagonal" and "leaves the s's alone". The matrices that result have the property that their entries depend only on the product $st$. A similar conjugation is possible whenever the Coxeter graph $\Gamma_M$ of $M$ is a forest ($\Gamma_M$ has vertices $X$ and an edge connecting $x_i$ and $x_j$ provided $m(i,j) \geq 3$). In these cases, the representations $\rho$ of $G_M$ is conjugate to a representation over the Laurent-polynomial ring $R[st,(st)^{-1}] \subseteq \Lambda$. In the case of $B_n$, the representation that results is the Burau representation.

In general, the question of the faithfulness of $\rho$ remains open. The only known cases seem to be those that follow easily from Theorem 2: $G_M$ is a direct product of rank 1 or 2 Artin groups (equivalently, $\Gamma_M$ is a disjoint union of vertices and pairs of vertices connected by an edge). Much effort has been devoted (unsuccessfully) to trying to determine whether or not the Burau representation of $B_4$ is faithful. One other case that might be worth investigating is $M$ defined by each $m(i,j) = \infty$, so that $G_M$ is a free group.

**REFERENCES**