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CLASS GROUPS OF RANK ONE SEMISIMPLE MONOIDS

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ABSTRACT. Let M be an irreducible, normal, algebraic monoid with unit group $Sl_2(k)\times k^*$, $Gl_2(k)$ or $PGl_2(k)\times k^*$. In [7] these monoids are classified numerically. In this paper we compute explicitly the divisor class group of each monoid. As a corollary we characterize the monoids with factorial coordinate algebra. All results are independent of the characteristic of k.

Introduction. In [7] the classification problem for semisimple rank one algebraic monoids is solved numerically. Associated with each semisimple monoid M, with unit group $G = Gl_2(k)$, $Sl_2(k) \times k^*$ or $PGl_2(k) \times k^*$, is its *polyhedral root system* (X, Φ, C), which is a complete and discriminating invariant of M. Furthermore, for each G there is a canonical one-to-one correspondence between the essentially distinct polyhedral root systems and the set of positive rational numbers [7]. Thus, each M is isomorphic to a unique M_r, $r \in \mathbf{Q}^+$.

The purpose of this paper is to interpret the correspondence $M \mapsto r$, $r \in \mathbf{Q}^+$, in terms of more conventional invariants of M; in particular, the divisor class group [2].

To state the main result let $M = M_r$, $r \in \mathbf{Q}^+$, be a semisimple monoid with G as above, and write $r = s/t$ where $(s,t) = 1$. Let $\mathrm{Cl}(M)$ be the divisor class group of M, and let $\mathbf{Z}_s$ denote the integers modulo s.

THEOREM. (a) *If* $G = Sl_2(k) \times k^*$ *then* $\mathrm{Cl}(M) \cong \mathbf{Z}_s$.
(b) *If* $G = Gl_2(k)$ *then*

$$\mathrm{Cl}(M) \cong \begin{cases} \mathbf{Z}_{2s} & \text{if } 2 \mid st, \\ \mathbf{Z}_s & \text{if } 2 \nmid st. \end{cases}$$

(c) *If* $G = PGl_2(k) \times k^*$ *then* $\mathrm{Cl}(M) \cong \mathbf{Z}_s \oplus \mathbf{Z}_2$.

One major technique in the proof is the *big cell* $U \subseteq M$, which was constructed in [7] for a different purpose. In this paper, it is used to reduce our problems to numerical questions about the polyhedral root system.

Background. An *algebraic monoid* is an affine algebraic variety M defined over the algebraically closed field k, together with an associative morphism $\mu\colon M \times M \to M$ and a two-sided unit $1 \in M$ for μ. We assume M is *irreducible* as a variety. In this case, $G = \{x \in G \mid x^{-1} \in M\} \subseteq M$ is open and dense in the Zariski topology [4]. $E(M) = \{e \in M \mid e^2 = e\} \subseteq M$ is the set of *idempotents* of M. A submonoid $Z \subseteq M$ is a *D-submonoid* if $Z = \overline{T}$, the Zariski closure of some torus $T \subseteq G$. By

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[**5**, Corollary 2], every idempotent e of M is an element of some D-submonoid Z of M. Note that every D-monoid Z is uniquely determined by its *character* monoid $X(Z) = \{\chi \in k[Z] \mid \chi\colon Z \to k \text{ is a morphism of monoids}\}$. An irreducible monoid M is *semisimple* if $0 \in M$, M is normal, and $\dim ZG = 1$, where ZG is the center of G [**7**]. In this paper, we are concerned mainly with semisimple monoids with unit group $Gl_2(k)$, $Sl_2(k) \times k^*$ or $PGl_2(k) \times k^*$. Such monoids will be called *semisimple rank one.*

Main results. If R is a Noetherian domain, integrally closed in its fraction field, we define the *divisor class group*, $\mathrm{Cl}(R)$ (as in [**2**]) to be the group of divisorial ideals of R modulo the group of principal ideals. If $R = k[X]$ for some normal affine variety X, we put $\mathrm{Cl}(X) = \mathrm{Cl}(R)$. Recall that any pure codimension one subvariety D of M determines an element $D \in \mathrm{Cl}(X)$. [**2**].

3.1 LEMMA. *Let M be a semisimple rank one monoid and let $D \in \mathrm{Cl}(M)$ be the divisor class of $M \backslash G \subseteq M$. Then*
(a) $0 \to \mathbf{Z} \cdot D \to \mathrm{Cl}(M) \to \mathrm{Cl}(G) \to 0$ *is exact.*
(b) $\mathrm{Cl}(M)$ *is finite.*

PROOF. (a) follows from Nagata's theorem [**2**], since by [**7**, Corollary 3.3], $M \backslash G$ is irreducible. Now by [**6**, Theorem 3.4], there exists $\chi\colon M \to k$ such that $\chi^{-1}(0) = M \backslash G$. Thus, in $\mathrm{Cl}(M)$, some multiple of D is zero, so that $\mathbf{Z} \cdot D$ is finite. Hence, using (a), $\mathrm{Cl}(M)$ is finite, since by [**3**], $\mathrm{Cl}(G) \cong \mathbf{Z}_2$ if $G = PGl_2(k) \times k^*$ and (0) otherwise. □

3.2 LEMMA. *Let M be as in* 3.1 *and let $X(M) = \{\chi\colon M \to k \mid \chi$ is a morphism of algebraic monoids$\}$. So $j\colon X(M) \subseteq X(G)$ by restriction. Then $X(M) \cong \mathbf{N}$ and $X(G) \cong \mathbf{Z}$ in such a way that j is the usual inclusion.*

PROOF. By [**6**], $X(M) \neq \varnothing$ so let $X'(M) = \{\chi \in X(G) \mid \chi^n \in X(M) \text{ some } n\}$. But then $X(M) = X'(M)$ since M is normal. It follows that $X(M)$, $X(G)$ and j are as stated. □

Let $\overline{T} \subseteq M$ be the Zariski closure in M of a maximal torus T of G. By [**8**] there exist unique V, $Y \in X(\overline{T})$ such that
(a) For all $\chi \in X(\overline{T})$ there exists $k, l, m \in \mathbf{N}$ such that $\chi^m = V^k Y^l$.
(b) If $\{g, h\} \subseteq X(\overline{T})$ satisfies (a) then $g = V^s Y^t$ and $h = V^u Y^v$ for appropriate $s, t, u, v, \in \mathbf{N}$.
$\{V, Y\}$ is the set of *fundamental generators.*

Let $\overline{T} = T \cup Te \cup Tf \cup \{0\}$, as in [**7**, §3]. Then $k[\overline{T}][V^{-1}]$ is the coordinate algebra of the open submonoid $T_e = T \cup Te$, and similarly for $k[\overline{T}][Y^{-1}]$ and $T_f = T \cup Tf$. By [**1**, Comment 3.3], or a direct argument, $T_e \cong k \times k^*$ (and similarly for T_f). In particular, $k[T_e]$ and $k[T_f]$ are UFDs, and so
(i) $\{\Psi \in k[T_e] | \Psi|_{T_e \backslash T} = 0\}$ is a principal ideal of $k[T_e]$, and
(ii) $X(T_e) \cong \mathbf{Z} \oplus \mathbf{N}$.

3.3 MAIN LEMMA. *Let $\overline{T}$ be as above, and let $\chi \in X(T_e)$ be the restriction to T_e of the generator $g \in X(M) \cong \mathbf{N}$ (as in* 3.2*). Write $\chi = U^s V^m \in X(T_e) = \{U^a V^b | a \in \mathbf{N}, b \in \mathbf{Z}\}$. Then the order of $D \in \mathrm{Cl}(M)$ (as in* 3.1*) is s.*

PROOF. Let $B = \{g \in G \mid ge = ege\} \subseteq G$. Then by [**7**, Proposition 2.4], B is a Borel subgroup and $B^- = \{g \in G \mid eg = ege\}$ is opposite to B relative to T.

Furthermore, by [**7**, Proposition 3.2],

$$\tau\colon U^- \times T_e \times U \to M, \qquad \tau(u,y,v) = uyv$$

is an open embedding, where U and U^- are the unipotent radicals of B and B^- respectively. Similarly $\nu\colon U \times T_f \times U^- \to M$, $\nu(u,y,v) = uyv$, is an open embedding. Let $\mathscr{V}_e$ =image(τ) and $\mathscr{V}_f$ = image(ν). Note that $\mathscr{V}_e$ and $\mathscr{V}_f$ are both *affine* open subsets of M, while $\mathrm{Cl}(M)$ is finite. It follows that there exist $r, s \in k[M]$ such that $k[\mathscr{V}_e] = k[M][1/r]$ and $k[\mathscr{V}_f] = k[M][1/s]$. Furthermore, height $(r,s) = 2$, since by [**7**, §2], $M\backslash\mathscr{V}_e$ and $M\backslash\mathscr{V}_f$ are both *irreducible* closed subsets of M. Now let

$$\mu = \{g \in k[M] | g|_{M\backslash G} = 0\}.$$

Then

$$o(D) = \inf\{s \in \mathbf{N} \mid \mu^{(s)} \text{ is principal}\}$$

where $\mu^{(s)}$ denotes the sth symbolic power of μ [**2**]. On the other hand, if $\mu^{(s)}$ is principal, then $\mu^{(s)} = (\chi)$ for some $\chi \in X(M)$, since these are the only principal ideals of $k[M]$ stable under both right and left translation by G. Thus, if $\chi \in X(M) \cong \mathbf{N}$ is the generator, then $o(D) = s$, where $(\chi) = \mu^{(s)}$. Finally, $\mathscr{V}_e \to M$ induces $k[M] \to k[\mathscr{V}_e] = k[M][1/r] \cong k[A^-, U, V, V^{-1}, A]$, where the last isomorphism results from the fact that $\mathscr{V}_e \cong k \times (k \times k^*) \times k$. Now $\mu \cdot k[\mathscr{V}_e] = (u)$ for some prime element $u \in k[M][1/r]$, since $k[\mathscr{V}_e]$ is a UFD. Thus, $(\chi)k[\mathscr{V}_e] = \mu^{(s)}k[\mathscr{V}_e] = (u)^{(s)} = (u^s)$. Hence, $\chi = u^s\varsigma$ for some unit $\varsigma \in k[\mathscr{V}_e]$. But then $\varsigma = V^m$, some m and thus, $\chi = U^sV^m \in k[T_e]$. □

3.4 COROLLARY. *Let* $\wp = \{\Psi \in k[\overline{T}] | \Psi|_{\overline{T}\backslash T} = 0\}$. *Then* μ *is principal iff* $\wp$ *is principal iff* $\mathrm{Cl}(M) = \mathrm{Cl}(G)$.

PROOF. μ is principal iff $\chi = UV^m \in k[T_e] = k[U, V, V^{-1}]$ iff $\chi = XY^m \in k[T_f] = k[X, Y, Y^{-1}]$. So μ is principal if and only if $\wp k[T_e]$ and $\wp k[T_f]$ are principal ideals generated by χ. But height $(V, Y) = 2$ as an ideal of $k[\overline{T}]$. Thus, $(\chi) = \wp$. □

3.5. Let M be a semisimple rank one monoid with unit group G and maximal torus $T \subseteq G$, and let $\overline{T} \subseteq M$. Then by [**7**, Theorem 5.3], $M = M_r$, for some $r \in \mathbf{Q}^+$ and by [**7**, §4.5],

(a) $X(\overline{T}) \cong X(r) = \{(b,a) \in X(T) = \mathbf{Z} \oplus \mathbf{Z} \mid |a/b| \leq r\} \cup \{(0,0)\}$ if $G = Sl_2(k) \times k^*$.

(b) $X(\overline{T}) \cong Y(r) = \{(b,a) \in X(T) = \mathbf{Z} \oplus \mathbf{Z} \mid |(a-b)/(a+b)| \leq r\} \cup \{(0,0)\}$ if $G = Gl_2(k)$.

(c) $X(\overline{T}) = X(r)$ if $G = PGl_2(k) \times k^*$.

Note. The "axes" have been interchanged in this paper. In cases (a) and (c) $\chi = (1,0) \in X(\overline{T})$ is the character that extends to the generator of $X(M) = \mathbf{N}$ (as in 3.2). In case (b), $\chi = (1,1)$.

3.6 LEMMA. (a) *Suppose* $m, n, a, b \in \mathbf{Z}$ *and* $mb - na = 1$. *If* $\alpha(n,m) + \beta(b,a) = (1,0)$ *then* $\beta = m$.

(b) *Suppose instead that* $\alpha(n,m) + \beta(b,a) = (1,1)$. *Then* $\beta = m - n$.

PROOF. Obvious.

3.7 THEOREM. *Let $M = M_r$, $r \in \mathbf{Q}^+$, be a semisimple, rank one, algebraic monoid, and let $D \in \mathrm{Cl}(M)$ be as in* 3.1. *Write $r = s/t$ where $(s,t) = 1$. Then $o(D) = s$, unless $G = Gl_2(k)$ and $2 \nmid st$, in which case $o(D) = 2s$.*

PROOF. Let $V, Y \in X(\overline{T})$ be the fundamental generators, and assume that $G = Gl_2(k)$ (the other cases are slightly easier). By 3.5(b), $V = (n,m)$, where $(m-n)/(m+n) = r$. Let $(b,a) \in X(\overline{T})$ such that $mb - na = 1$. By 3.6(b), if $\alpha(n,m) + \beta(b,a) = (1,1)$ then $\beta = m - n$, and so by 3.3, $o(D) = m - n$. Now if m and n are not both odd, then $(m-n, m+n) = 1$, and so $m - n = s$ and $m + n = t$. If m and n are both odd then $(m-n, m+n) = 2$ and so $(m-n)/2 = s$ and $(m+n)/2 = t$. So in this case, $m - n = 2s$ and $m + n = 2t$. □

3.8 THEOREM. *Let $M = M_r$ be a semisimple rank one monoid with unit group G. Write $r = s/t$, $(s,t) = 1$.*

(a) *If $G = Sl_2(k) \times k^*$ then* $\mathrm{Cl}(M) \cong \mathbf{Z}_s$.

(b) *If $G = Gl_2(k)$ then*

$$\mathrm{Cl}(M) \cong \begin{cases} \mathbf{Z}_{2s} & \text{if } 2 \mid st, \\ \mathbf{Z}_s & \text{if } 2 \nmid st. \end{cases}$$

(c) *If $G = PGl_2(k) \times k^*$ then* $\mathrm{Cl}(M) \cong \mathbf{Z}_s \oplus \mathbf{Z}_2$.

PROOF. Cases (a) and (b) follow from 3.1 and 3.7 since $\mathrm{Cl}(G) = (0)$ in these cases [**3**]. For (c) we distinguish two cases. If st is odd then by 3.1 and 3.7, $\mathrm{Cl}(M)$ is an extension of a cyclic group of odd order by $\mathbf{Z}_2$. But any such extension splits. If st is even consider the map $\Psi\colon Gl_2(k) \to PGl_2(k) \times k^*$ of degree two (factor out the central subgroup of $Gl_2(k)$ of order two). Let $\Psi\colon M' \to M$ be the normalization of M along Ψ (see [**8**, Lemma 7.1.1]). It follows from 3.5(b) that M' also corresponds to r for the group $Gl_2(k)$. It is also easy to check that $\Psi^*\colon X(M) \to X(M')$ is bijective. Thus, by 3.3 $\mathrm{Cl}(\Psi)\colon \mathbf{Z}\cdot D \to \mathbf{Z}\cdot D'$ is injective and by 3.7 it is surjective since these groups have the same order. Thus we have the desired splitting

$$\begin{array}{ccccccccc} 0 & \to & \mathbf{Z}\cdot D & \to & \mathrm{Cl}(M) & \to & \mathbf{Z}_2 & \to & 0 \\ & & & \cong\searrow & \downarrow & & & & \\ & & & & \mathrm{Cl}(M') & & & & \end{array}$$

This concludes the proof.

3.9 COROLLARY. *Let $M = M_r$ be a semisimple rank one monoid with unit group G. Then the following are equivalent.*

(a) *$k[M]$ is factorial.*

(b) *$G = Sl_2(k) \times k^*$ and $r = 1/t$, or $G = Gl_2(k)$ and $r = 1/t$ with $2 \nmid t$.*

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