PLURISUBHARMONIC FUNCTIONS OUTSIDE COMPACT SETS

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(Communicated by Irwin Kra)

ABSTRACT. We study the Monge-Ampère mass of plurisubharmonic functions defined outside a compact set.

1. Introduction. Let \( \Omega \) be an open and bounded pseudoconvex set in \( \mathbb{C}^n \), \( n \geq 2 \), and \( K \) a compact subset of \( \Omega \) so that \( \Omega \setminus K \) is connected. By the Hartogs extension theorem, every analytic function on \( \Omega \setminus K \) extends to \( \Omega \). When it comes to plurisubharmonic functions the situation is different.

The purpose of this note is to study the Monge-Ampère mass \( \int_{\Omega \setminus K} (dd^c u)^n \), where \( u \in \text{PSH} \cap L^\infty_{\text{loc}}(\Omega \setminus K) \) and \( K \subseteq \Omega' \subseteq \Omega \).

The case when \( u \in \text{PSH} \cap C^\infty(\Omega \setminus K) \) has already been studied by Fornaess and Sibony [3] and Griffith [4].

2. Estimates.

Proposition. Assume that \( \varphi \in \text{PSH} \cap L^\infty_{\text{loc}}(\Omega \setminus K) \), where \( K \) is a removable singularity set for the plurisubharmonic functions (cf. [2]). Then \( \int_{\Omega' \setminus K} (dd^c \varphi)^n < +\infty \).

Proof. By Bedford and Taylor [1], \( (dd^c \varphi)^n \) is a well-defined positive measure on \( \Omega' \setminus K \). Since \( \varphi \) extends to a plurisubharmonic function on \( \Omega \), we can choose a sequence \( \varphi_j \in \text{PSH} \cap C^\infty \) near \( \Omega' \) such that \( \varphi_j \searrow \varphi \), \( j \to +\infty \).

Let \( W = \{ z \in \Omega' \mid d(z, K) > \frac{1}{n} d(\partial \Omega', K) \} \). Then \( \delta = \inf_{z \in W} \varphi(z) > -\infty \), and if we put \( \psi_j = \sup(\varphi_j, \delta) \), then \( \psi_j = \varphi_j \) on \( W \). So \( \int_{\Omega'} (dd^c \psi_j)^n = \int_{\Omega'} (dd^c \varphi_j)^n \) by Stokes' theorem. Therefore,

\[
\int_{\Omega' \setminus K} (dd^c \varphi)^n \leq \lim_{j \to +\infty} \int_{\Omega'} (dd^c \varphi_j)^n = \lim_{j \to +\infty} \int_{\Omega'} (dd^c \psi_j)^n = \int_{\Omega'} (dd^c \sup(\varphi, \delta))^n < +\infty
\]

since \( \sup(\varphi, \delta) \in \text{PSH} \cap L^\infty_{\text{loc}}(\Omega) \).

The next theorem is a generalization of Theorem 2.2 in Fornaess and Sibony [3].

Theorem. Let \( \psi \in \text{PSH} \cap C^2(\Omega) \) and assume that \( \Omega' = \{ \psi < 1 \} \) and that \( K = \{ \psi \leq s \} \) for an \( s \), \( 0 < s < 1 \). If \( \varphi \in \text{PSH} \cap L^\infty_{\text{loc}}(\Omega \setminus K) \), then

\[
\int_{\{ s < \psi < 1 \}} |\varphi|(dd^c \varphi)^{n-1} \land dd^c \psi < +\infty
\]
and

\[ \int_{\{s<\psi<1\}} (\psi - s)(ddc^c \varphi)^n < +\infty. \]

**Proof.** Since \( \Omega \) is pseudoconvex and \( \psi \) plurisubharmonic, there is, to every \( z_0 \in \{ s < \psi < 1 \} \), an analytic function such that \( f(z) \neq f(z_0) \) \( \forall z \in K \). If we restrict \( \varphi \) to \( \{ f(z) = f(z_0) \} \) and apply the maximum principle, we conclude that \( \varphi \) is uniformly bounded above on \( \Omega \setminus K \). It is therefore no restriction to assume that \( \varphi \leq 0 \).

Let \( s < r < 1 \). Then, by Stokes' theorem we have

\[
\begin{align*}
\int_{\{r<\psi<1\}} -\varphi(ddc^c \psi) \wedge (ddc^c \varphi)^{n-1} + \int_{\{r<\psi<1\}} (\psi - r)(ddc^c \varphi)^n \\
\leq \lim_{t \searrow r} \left[ \int_{\{r<\psi<1\}} -\varphi dd^c \max(\psi, t) \wedge (ddc^c \varphi)^{n-1} + \int_{\{r<\psi<1\}} (\psi - r)(ddc^c \varphi)^n \right] \\
= \lim_{t \searrow r} \left[ \int_{\{\psi=1\}} -\varphi dd^c (\psi, t) \wedge (ddc^c \varphi)^{n-1} \\
+ \int_{\{\psi=1\}} d \max(\psi, t) \wedge dd^c \varphi \wedge (ddc^c \varphi)^{n-1} + \int_{\{r<\psi<1\}} (\psi - r)(ddc^c \varphi)^n \right] \\
= \lim_{t \searrow r} \left[ \int_{\{\psi=1\}} -\varphi dd^c (\psi, t) \wedge (ddc^c \varphi)^{n-1} \\
+ \int_{\{\psi=1\}} (\max(\psi, t) - r) dd^c \varphi \wedge (ddc^c \varphi)^{n-1} \\
- \int_{\{\psi=r\}} (\max(\psi, t) - r) dd^c \varphi \wedge (ddc^c \varphi)^{n-1} \\
+ \int_{\{r<\psi<1\}} (\psi - \max(\psi, t))(ddc^c \varphi)^n \right] \\
= \int_{\{\psi=1\}} -\varphi dd^c \psi \wedge (ddc^c \varphi)^{n-1} + \int_{\{\psi=1\}} (\psi - r) dd^c \varphi \wedge (ddc^c \varphi)^{n-1}.
\end{align*}
\]

Since the right-hand side is uniformly bounded in \( r, s < r < 1 \), and since \( \varphi \leq 0 \), we get

\[
\int_{\{s<\psi<1\}} |\varphi|dd^c \psi \wedge (ddc^c \varphi)^{n-1} + \int_{\{s<\psi<1\}} (\psi - s)(ddc^c \varphi)^n < +\infty,
\]

where each member is nonnegative.

3. **An example.** The following example shows that the convergence factor \( \psi - s \) really is needed in the Theorem. There are functions \( \varphi \) and \( \psi \) such that \( \int_{\{s<\psi<1\}} (ddc^c \varphi)^n = +\infty \), where \( \varphi \) can be taken to be plurisubharmonic and bounded.

Define \( u(z) = -\sqrt{|z|^2 - \frac{1}{2}} \); then \( u \) is subharmonic on \( \{ z \in \mathbb{C}; \frac{1}{2} < |z|^2 < 1 \} \), \(-1/\sqrt{2} < u < 0 \), and \( \Delta u = (1 - |z|^2)/(|z|^2 - \frac{1}{2})^{3/2} \).
Define $\psi(z, w) = |z|^2 + \frac{2}{3}|w|^8$, and let $0 < \eta < \frac{1}{20}$, and put

$$\varphi_\eta(z, w) = \max \left[ -\frac{1}{2} \sqrt{(1 + \eta)^2 |z|^2 - \frac{1}{2} + |w|^2}, |w|^2 + (1 + \eta)|w|^4 - \sqrt{\frac{3}{2}(\eta + \eta^2/2)} \right].$$

Then $\varphi_\eta$ is plurisubharmonic on $\{\frac{1}{2} < \psi < (20/21)^2\}$ for, if $(1 + \eta)^2 |z|^2 \leq \frac{1}{2}$, we have, when $\psi(z, w) > \frac{1}{2}$,

$$\frac{2}{3} |w|^8 > \frac{1}{2} - |z|^2 > \frac{1}{2}(1 - (1 + \eta)^{-2}) = \frac{1}{2}(2\eta + \eta^2)/(1 + \eta)^2 = (\eta + \eta^2/2)(1 + \eta)^2$$

so

$$(1 + \eta)|w|^4 > \sqrt{\frac{3}{2}(\eta + \eta^2/2)}. \quad \text{Assume now that } \frac{1}{2} < |z|^2 + \frac{2}{3}|w|^8 < (20/21)^2 \text{ and that}$$

$$\sqrt{\frac{3}{5} \frac{\sqrt{\eta + \eta^2/2}}{1 + \eta}} < |w|^4 < \sqrt{\frac{2}{3} \frac{39}{40} \frac{\sqrt{\eta + \eta^2/2}}{1 + \eta}}.$$ 

Then

$$\varphi_\eta = -\frac{1}{2} \sqrt{(1 + \eta)^2 |z|^2 - \frac{1}{2} + |w|^2}$$

if

$$-\frac{1}{2} \sqrt{(1 + \eta)^2 |z|^2 - \frac{1}{2}} > (1 + \eta)|w|^4 - \sqrt{\frac{3}{2}(\eta + \eta^2/2)}$$

$$\Rightarrow (1 + \eta)^2 |z|^2 < \frac{1}{2} + 4 \left( \sqrt{\frac{3}{2}(\eta + \eta^2/2)} - (1 + \eta)|w|^4 \right)^2,$$

which holds if

$$\left( \frac{1}{2} - \frac{2}{3}|w|^8 \right)(1 + \eta)^2 < (1 + \eta)^2 |z|^2 < \frac{1}{2} + 4 \left( \sqrt{\frac{3}{2}(\eta + \eta^2/2)} - (1 + \eta)|w|^4 \right)^2,$$

which is a well-defined domain if

$$\eta + \eta^2/2 - \frac{2}{3}|w|^8(1 + \eta)^2$$

$$< 6(\eta + \eta^2/2) - 8 \sqrt{\frac{3}{2}(\eta + \eta^2/2)(1 + \eta)|w|^4 + 4(1 + \eta)^2|w|^8}$$

$$\Rightarrow 0 < 5(\eta + \eta^2/2) + \frac{16}{3}(1 + \eta)^2|w|^8 - 8 \sqrt{\frac{3}{2}(\eta + \eta^2/2)(1 + \eta)|w|^4},$$

which holds with $|w|^4$ in the interval above—the right-hand side is then strictly larger than

$$5 \left( \frac{\eta}{2} + \frac{\eta^2}{2} \right) + \frac{14}{3} (1 + \eta)^2 \frac{3}{5} \eta + \frac{\eta^2}{2} = 8 \cdot \frac{39}{40} \left( \eta + \frac{\eta^2}{2} \right) = 0.$$ 

Therefore,

$$\int_{\frac{1}{2} < |z|^2 + \frac{2}{3}|w|^8 < (20/21)^2} (dd^c \varphi_\eta)^2 \geq c \int_{A_\eta} \frac{dz \, dw}{((1 + \eta)^2 |z|^2 - \frac{1}{2})^{3/2}},$$

where

$$A_\eta = \left\{ \left( \frac{1}{2} - \frac{2}{3}|w|^8 \right)(1 + \eta)^2 < (1 + \eta)^2 |z|^2 < \frac{1}{2} + 4 \left( \sqrt{\frac{3}{2}(\eta + \eta^2/2)} - (1 + \eta)|w|^4 \right)^2, \frac{\sqrt{\eta + \eta^2/2}}{1 + \eta} < |w|^4 < \frac{\beta \sqrt{\eta + \eta^2/2}}{1 + \eta} \right\},$$

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and where we have chosen \( \frac{\sqrt{3}}{5} < \alpha < \beta < \frac{39}{40} \frac{\sqrt{2}}{3} \) and where \( c \) is a strictly positive constant. The right-hand side is not smaller than

\[
c \left\{ \left( \alpha^2 - \frac{3}{5} \right) + \frac{39}{40} \sqrt{\frac{2}{3}} - \beta \right\} \int \frac{(\eta + \eta^2/2) \, dw}{2^3 \left( \frac{\sqrt{3}}{2} (\eta + \eta^2/2) - (1 + \eta)|w|^4 \right)^3}
\]

\[
\frac{\sqrt{\frac{3}{5}} (\eta + \eta^2/2)}{1 + \eta} < |w|^4 < \frac{\sqrt{\frac{2}{3}} \frac{39}{40} \sqrt{\eta + \eta^2/2}}{1 + \eta}
\]

\[
\geq d (\eta + \eta^2/2)^{1-3/2+1/4}
\]

\[
= \frac{d}{(\eta + \eta^2/2)^{1/4}} \to +\infty, \quad \eta \to 0 \ (d \text{ pos. const.}).
\]

Furthermore, \(-\frac{1}{2} < \varphi_\eta \leq 2\) so if we put

\[
\varphi = \sum_{j=20}^{\infty} \frac{1}{j^2} \varphi_1^{1/4},
\]

then

\[
\varphi \in \text{PSH} \cap L^\infty \left( \frac{1}{2} < |z|^2 + \frac{2}{3} |w|^8 < \left( \frac{20^4}{(20^4 + 1)} \right)^2 \right)
\]

and

\[
\int_{\frac{1}{4} < |z|^2 + \frac{2}{3} |w|^8 < (20/21)^2} (dd^c \varphi)^2 = +\infty.
\]

**Remark.** It is possible to modify the \( \varphi_\eta \)'s to get the function \( \varphi \in C^\infty \) on \( \Omega \setminus \{ w = 0 \} \). We do not know if one can modify \( \varphi \) to be \( C^\infty \) on \( \Omega \) but still have \( \int_{\Omega \setminus K} (dd^c \varphi)^2 = +\infty \).

**References**


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