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PLURISUBHARMONIC FUNCTIONS OUTSIDE COMPACT SETS

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ABSTRACT. We study the Monge-Ampère mass of plurisubharmonic functions defined outside a compact set.

1. Introduction. Let Ω be an open and bounded pseudoconvex set in $\mathbf{C}^n$, $n \geq 2$, and K a compact subset of Ω so that $\Omega \backslash K$ is connected. By the Hartogs extension theorem, every analytic function on $\Omega \backslash K$ extends to Ω. When it comes to plurisubharmonic functions the situation is different.

The purpose of this note is to study the Monge-Ampère mass $\int_{\Omega' \backslash K} (dd^c u)^n$, where $u \in \mathrm{PSH} \cap L^\infty_{\mathrm{loc}}(\Omega \backslash K)$ and $K \Subset \Omega' \Subset \Omega$.

The case when $u \in \mathrm{PSH} \cap C^\infty(\Omega \backslash K)$ has already been studied by Fornaess and Sibony [**3**] and Griffith [**4**].

2. Estimates.

PROPOSITION. *Assume that $\varphi \in \mathrm{PSH} \cap L^\infty_{\mathrm{loc}}(\Omega \backslash K)$, where K is a removable singularity set for the plurisubharmonic functions (cf.* [**2**]*). Then $\int_{\Omega' \backslash K} (dd^c \varphi)^n < +\infty$.*

PROOF. By Bedford and Taylor [**1**], $(dd^c \varphi)^n$ is a well-defined positive measure on $\Omega \backslash K$. Since φ extends to a plurisubharmonic function on Ω, we can choose a sequence $\varphi_j \in \mathrm{PSH} \cap C^\infty$ near $\overline{\Omega'}$ such that $\varphi_j \searrow \varphi$, $j \to +\infty$.

Let $W = \{z \in \Omega',\ d(z, K) > \frac{1}{2} d(\partial \Omega', K)\}$. Then $\delta = \inf_{z \in W} \varphi(z) > -\infty$, and if we put $\psi_j = \sup(\varphi_j, \delta)$, then $\psi_j = \varphi_j$ on W. So $\int_{\Omega'} (dd^c \psi_j)^n = \int_{\Omega'} (dd^c \varphi_j)^n$ by Stokes' theorem. Therefore,

$$\begin{aligned}\int_{\Omega' \backslash K} (dd^c \varphi)^n &\leq \lim_{j \to +\infty} \int_{\Omega'} (dd^c \varphi_j) \\ &= \lim_{j \to +\infty} \int_{\Omega'} (dd^c \psi_j)^n \\ &= \int_{\Omega'} (dd^c \ \sup(\varphi, \delta))^n < +\infty\end{aligned}$$

since $\sup(\varphi, \delta) \in \mathrm{PSH} \cap L^\infty_{\mathrm{loc}}(\Omega)$.

The next theorem is a generalization of Theorem 2.2 in Fornaess and Sibony [**3**].

THEOREM. *Let $\psi \in \mathrm{PSH} \cap C^2(\Omega)$ and assume that $\Omega' = \{\psi < 1\}$ and that $K = \{\psi \leq s\}$ for an s, $0 < s < 1$. If $\varphi \in \mathrm{PSH} \cap L^\infty_{\mathrm{loc}}(\Omega \backslash K)$, then*

$$\int_{\{s < \psi < 1\}} |\varphi| (dd^c \varphi)^{n-1} \wedge dd^c \psi < +\infty$$

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and

$$\int_{\{s<\psi<1\}} (\psi - s)(dd^c\varphi)^n < +\infty.$$

PROOF. Since Ω is pseudoconvex and ψ plurisubharmonic, there is, to every $z_0 \in \{s < \psi < 1\}$, an analytic function such that $f(z) \neq f(z_0)$ $\forall z \in K$. If we restrict φ to $\{f(z) = f(z_0)\}$ and apply the maximum principle, we conclude that φ is uniformly bounded above on $\overline{\Omega'\backslash K}$. It is therefore no restriction to assume that $\varphi \leq 0$.

Let $s < r < 1$. Then, by Stokes' theorem we have

$$\begin{aligned}
&\int_{\{r<\psi<1\}} -\varphi(dd^c\psi) \wedge (dd^c\varphi)^{n-1} + \int_{\{r<\psi<1\}} (\psi - r)(dd^c\varphi)^n \\
&\quad \leq \varliminf_{t\searrow r} \int_{\{r<\psi<1\}} -\varphi\, dd^c(\max\ \psi, t) \wedge (dd^c\varphi)^{n-1} + \int_{\{r<\psi<1\}} (\psi - r)(dd^c\varphi)^n \\
&\quad = \varliminf_{t\searrow r} \Bigg[\int_{\{\psi=1\}} -\varphi\, d^c(\max\psi, t) \wedge (dd^c\varphi)^{n-1} \\
&\qquad + \int_{\{r<\psi<1\}} d\max(\psi, t) \wedge d^c\varphi \wedge (dd^c\varphi)^{n-1} + \int_{\{r<\psi<1\}} (\psi - r)(dd^c\varphi)^n\Bigg] \\
&\quad = \varliminf_{t\searrow r} \Bigg[\int_{\{\psi=1\}} -\varphi\, d^c(\psi, t) \wedge (dd^c\varphi)^{n-1} \\
&\qquad + \int_{\{\psi=1\}} (\max(\psi, t) - r)\, d^c\varphi \wedge (dd^c\varphi)^{n-1} \\
&\qquad - \int_{\{\psi=r\}} (\max(\psi, t) - r)\, d^c\varphi \wedge (dd^c\varphi)^{n-1} \\
&\qquad\qquad + \int_{\{r<\psi<1\}} (\psi - \max(\psi, t))(dd^c\varphi)^n\Bigg] \\
&\quad = \int_{\{\psi=1\}} -\varphi\, d^c\psi \wedge (dd^c\varphi)^{n-1} + \int_{\{\psi=1\}} (\psi - r)\, d^c\varphi \wedge (dd^c\varphi)^{n-1}.
\end{aligned}$$

Since the right-hand side is uniformly bounded in r, $s < r < 1$, and since $\varphi \leq 0$, we get

$$\int_{\{s<\psi<1\}} |\varphi| dd^c\psi \wedge (dd^c\varphi)^{n-1} + \int_{\{s<\psi<1\}} (\psi - s)(dd^c\varphi)^n < +\infty,$$

where each member is nonnegative.

3. An example. The following example shows that the convergence factor $\psi - s$ really is needed in the Theorem. There are functions φ and ψ such that $\int_{\{s<\psi<1\}} (dd^c\varphi)^n = +\infty$, where φ can be taken to be plurisubharmonic and bounded.

Define $u(z) = -\sqrt{|z|^2 - \frac{1}{2}}$; then u is subharmonic on $\{z \in \mathbf{C}; \frac{1}{2} < |z|^2 < 1\}$, $-1/\sqrt{2} < u < 0$, and $\Delta u = (1 - |z|^2)/(|z|^2 - \frac{1}{2})^{3/2}$.

Define $\psi(z,w) = |z|^2 + \frac{2}{3}|w|^8$, and let $0 < \eta < \frac{1}{20}$, and put

$$\varphi_\eta(z,w) = \max\left[-\tfrac{1}{2}\sqrt{(1+\eta)^2|z|^2 - \tfrac{1}{2}} + |w|^2, |w|^2 + (1+\eta)|w|^4 - \sqrt{\tfrac{3}{2}(\eta+\eta^2/2)}\right].$$

Then φ_η is plurisubharmonic on $\{\frac{1}{2} < \psi < (20/21)^2\}$ for, if $(1+\eta)^2|z|^2 \le \frac{1}{2}$, we have, when $\psi(z,w) > \frac{1}{2}$,

$$\tfrac{2}{3}|w|^8 > \tfrac{1}{2} - |z|^2 > \tfrac{1}{2}(1-(1+\eta)^{-2}) = \tfrac{1}{2}(2\eta+\eta^2)/(1+\eta)^2 = (\eta+\eta^2/2)(1+\eta)^2$$

so

$$(1+\eta)|w|^4 > \sqrt{\tfrac{3}{2}(\eta+\eta^2/2)}.$$

Assume now that $\frac{1}{2} < |z|^2 + \frac{2}{3}|w|^8 < (20/21)^2$ and that

$$\sqrt{\frac{3}{5}}\,\frac{\sqrt{\eta+\eta^2/2}}{1+\eta} < |w|^4 < \sqrt{\frac{2}{3}}\,\frac{39}{40}\,\frac{\sqrt{\eta+\eta^2/2}}{1+\eta}.$$

Then

$$\varphi_\eta = -\tfrac{1}{2}\sqrt{(1+\eta)^2|z|^2 - \tfrac{1}{2}} + |w|^2$$

if

$$-\tfrac{1}{2}\sqrt{(1+\eta)^2|z|^2 - \tfrac{1}{2}} > (1+\eta)|w|^4 - \sqrt{\tfrac{3}{2}(\eta+\eta^2/2)}$$
$$\Leftrightarrow (1+\eta)^2|z|^2 < \tfrac{1}{2} + 4\left(\sqrt{\tfrac{3}{2}(\eta+\eta^2/2)} - (1+\eta)|w|^4\right)^2,$$

which holds if

$$(\tfrac{1}{2} - \tfrac{2}{3}|w|^8)(1+\eta)^2 < (1+\eta)^2|z|^2 < \tfrac{1}{2} + 4\left(\sqrt{\tfrac{3}{2}(\eta+\eta^2/2)} - (1+\eta)|w|^4\right)^2$$

which is a well-defined domain if

$$\begin{aligned}
&\eta + \eta^2/2 - \tfrac{2}{3}|w|^8(1+\eta)^2 \\
&\qquad < 6(\eta+\eta^2/2) - 8\sqrt{\tfrac{3}{2}(\eta+\eta^2/2)}(1+\eta)|w|^4 + 4(1+\eta)^2|w|^8 \\
&\qquad \Leftrightarrow 0 < 5(\eta+\eta^2/2) + \tfrac{14}{3}(1+\eta)^2|w|^8 - 8\sqrt{\tfrac{3}{2}}\sqrt{\eta+\eta^2/2}(1+\eta)|w|^4,
\end{aligned}$$

which holds with $|w|^4$ in the interval above—the right-hand side is then strictly larger than

$$5\left(\eta + \frac{\eta^2}{2}\right) + \frac{14}{3}(1+\eta)^2\frac{3}{5}\,\frac{\eta+\eta^2/2}{(1+\eta)^2} - 8\cdot\frac{39}{40}\left(\eta+\frac{\eta^2}{2}\right) = 0.$$

Therefore,

$$\int_{\frac{1}{2}<|z|^2+\frac{2}{3}|w|^8<(20/21)^2} (dd^c\varphi_\eta)^2 \ge c\int_{A_\eta} \frac{dz\,dw}{((1+\eta)^2|z|^2 - \frac{1}{2})^{3/2}},$$

where

$$A_\eta = \begin{cases} (\frac{1}{2} - \frac{2}{3}|w|^8)(1+\eta)^2 < (1+\eta)^2|z|^2 < \frac{1}{2} + 4\left(\sqrt{\frac{3}{2}(\eta+\eta^2/2)} - (1+\eta)|w|^4\right)^2, \\ \dfrac{\sqrt{\eta+\eta^2/2}}{1+\eta}\alpha < |w|^4 < \beta\dfrac{\sqrt{\eta+\eta^2/2}}{1+\eta}, \end{cases}$$

and where we have chosen $\sqrt{\frac{3}{5}} < \alpha < \beta < \frac{39}{40}\sqrt{\frac{2}{3}}$ and where c is a strictly positive constant. The right-hand side is not smaller than

$$c\left\{\left(\alpha^2 - \frac{3}{5}\right) + \frac{39}{40}\sqrt{\frac{2}{3}} - \beta\right\} \int \frac{(\eta + \eta^2/2)\,dw}{2^3(\sqrt{\frac{3}{2}(\eta+\eta^2/2)} - (1+\eta)|w|^4)^3}$$

$$\begin{aligned} \frac{\sqrt{\frac{3}{5}(\eta+\eta^2/2)}}{1+\eta} < |w|^4 &< \sqrt{\frac{2}{3}}\frac{39}{40}\frac{\sqrt{\eta+\eta^2/2}}{1+\eta} \\ &\geq d(\eta+\eta^2/2)^{1-3/2+1/4} \\ &= \frac{d}{(\eta+\eta^2/2)^{1/4}} \to +\infty, \quad \eta \to 0 \ (d \text{ pos. const.}). \end{aligned}$$

Furthermore, $-\frac{1}{2} < \varphi_\eta \leq 2$ so if we put

$$\varphi = \sum_{j=20}^{\infty} \frac{1}{j^2}\varphi_{1/j^4},$$

then

$$\varphi \in \mathrm{PSH} \cap L^\infty\left(\tfrac{1}{2} < |z|^2 + \tfrac{2}{3}|w|^8 < \left(20^4/(20^4+1)\right)^2\right)$$

and

$$\int_{\frac{1}{2}<|z|^2+\frac{2}{3}|w|^8<(20/21)^2} (dd^c\varphi)^2 = +\infty.$$

REMARK. It is possible to modify the φ_η's to get the function φ C^∞ on $\Omega\backslash\{w=0\}$. We do not know if one can modify φ to be C^∞ on Ω but still have $\int_{\Omega\backslash K}(dd^c\varphi)^2 = +\infty$.

REFERENCES

1. E. Bedford and B. A. Taylor, *A new capacity for plurisubharmonic functions*, Acta Math. **199** (1982), 1–40.
2. U. Cegrell, *Removable singularity sets for plurisubharmonic functions and related problems*, Proc. London Math. Soc. (3) **36** (1978), 310–336.
3. J. E. Fornaess and N. Sibony, *Plurisubharmonic functions on ring domains*, Complex Analysis (S. G. Krantz, Ed.), Lecture Notes in Math., vol. 1268, Springer, Berlin and New York.
4. P. Griffith, *Two theorems on extension of holomorphic mappings*, Invent. Math. **14** (1971), 27–62.

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