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SMALL SYSTEMS CONVERGENCE AND METRIZABILITY

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ABSTRACT. The abstract notion of convergence of functions with respect to a small system has its roots in the concept of convergence of functions in measure. The second author has shown that convergence of functions with respect to a small system generates a Fréchet topology. In this paper we show that convergence with respect to a small system is equivalent to convergence with respect to a certain complete pseudometric (or metric if we consider equivalence classes of functions with respect to the small system).

1. Small systems. We begin with some definitions which are similar to those in [**1**, **7**, p. 491, and **8**].

1.1. DEFINITION. Let $\{\mathcal{N}_r\}$ be a decreasing sequence of nonempty families of subsets of X such that for each positive integer r, there exists a sequence $\{k_i\}$ of positive integers such that $E_i \in \mathcal{N}_{k_i}$ $(i = 1, 2, \dots)$ implies $\bigcup_{i=1}^{\infty} E_i \in \mathcal{N}_r$. Such a sequence $\{\mathcal{N}_r\}$ will be called a *small system* on X.

1.2. DEFINITION. Let $\{\mathcal{N}_r\}$ be a decreasing sequence of nonempty families of subsets of X such that for each positive integer r,

(i) $E_i \in \mathcal{N}_i$ $(i = r+1, r+2, \dots)$ implies $\bigcup_{i=r+1}^{\infty} E_i \in \mathcal{N}_r$, and

(ii) $E, F \in \mathcal{N}_{r+1}$ implies $E \cup F \in \mathcal{N}_r$.

The sequence $\{\mathcal{N}_r\}$ will be called a *strong small system* on X.

Evidently, every strong small system is a small system. We shall see later in Theorem 1.5 that for all practical purposes, every small system can be replaced by a *strong* small system.

1.3. EXAMPLES. For each of the following, $\{\mathcal{N}_r\}$ is an example of a small system on the given set X:

(i) Let $(X, \mathcal{S}, \mu)$ be a measure space and for each r, let

$$\mathcal{N}_r = \{E \in \mathcal{S}\colon\ \mu(E) \le 2^{-r}\}.$$

(ii) Let μ be a Borel measure on a topological space X. For each r, let

$$\mathcal{N}_r = \{E\colon\ E \text{ is open and } \mu(E) \le 1/r\}.$$

(iii) Let (X, d) be a metric space and $x_0 \in X$. For each r, let

$$\mathcal{N}_r = \{B(x_0, s)\colon\ 0 < s < 1/r\},$$

where $B(x_0, s) = \{x \in X\colon d(x_0, x) < s\}$.

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(iv) Let X be any nonempty set, and let $\{E_n\}$ be a decreasing sequence of subsets of X. For each r, let

$$\mathcal{N}_r = \{E_n\colon\ n \geq r\}.$$

(v) Let X be the reals and Q be the rationals. For each r, let

$$\mathcal{N}_r = \{E \subset Q\colon p/q \in E \text{ and } p/q \text{ in reduced form implies } q \geq r\}.$$

(vi) Let X be the reals. For each r, let

$$\mathcal{N}_r = \{[0,s) \cup E\colon\ 0 < s < 1/r \text{ and } E \text{ is meager}\}.$$

(vii) Let X be the reals. For each r, let

$$\mathcal{N}_r = \{E\colon \text{ Hausdorff dimension of } E \text{ is less than } 1/r\}.$$

(For a definition of Hausdorff dimension, see [**4**, p. 107 or **6**, p. 134].)

The small systems given in examples (i), (iv), (v), and (vii) are strong small systems. Obvious modifications in the other examples would make them strong small systems also.

1.4. DEFINITION. Let $\{\mathcal{N}_r\}$ and $\{\mathcal{M}_r\}$ be small systems on X. We say that $\{\mathcal{N}_r\}$ and $\{\mathcal{M}_r\}$ are *similar* if and only if for each positive integer r, there exist positive integers $s(r)$ and $t(r)$ such that $\mathcal{N}_{s(r)} \subset \mathcal{M}_r$ and $\mathcal{M}_{t(r)} \subset \mathcal{N}_r$.

1.5. THEOREM. *Let $\{\mathcal{N}_r\}$ be a small system on X. Then there exists an increasing sequence of positive integers $\{s(r)\}$ such that $\{\mathcal{N}_{s(r)}\}$ is a strong small system. If for each positive integer r, $\mathcal{M}_r = \mathcal{N}_{s(r)}$, then the strong small system $\{\mathcal{M}_r\}$ is clearly similar to $\{\mathcal{N}_r\}$.*

PROOF. Let $k_{1,i} = i$ for each positive integer i, and let $s(1) = 1$. By induction, define $\{k_{r+1,i}\}$ and $s(r+1)$ as follows: Choose for each positive integer r, an increasing sequence $\{k_{r+1,i}\}$ of positive integers such that $E_i \in \mathcal{N}_{k_{r+1,i}}$ $(i = 1, 2, \dots)$ implies $\bigcup_{i=1}^{\infty} E_i \in \mathcal{N}_{s(r)}$ and such that $k_{r,i} \leq k_{r+1,i}$ for each positive integer i. (The latter requirement can be achieved by dropping to a subsequence of $\{k_{r+1,i}\}$ if necessary.) Let $s(r+1) = k_{r+1,r+1}$ for each positive integer r.

It is easy to see that $E_j \in \mathcal{N}_{s(j)} = \mathcal{N}_{k_{j,j}}$ $(j = r+1, r+2, \dots)$ implies $\bigcup_{i=r+1}^{\infty}\{E_i\} \in \mathcal{N}_{s(r)}$, so that (i) holds in the definition of strong small system. If $E, F \in \mathcal{N}_{s(r+1)}$, then $E \in \mathcal{N}_{k_{r+1,r+1}} \subset \mathcal{N}_{k_{r+1,r}}$ and $F \in \mathcal{N}_{k_{r+1,r+1}}$. Hence, $E \cup F \in \mathcal{N}_{s(r)}$, so that (ii) holds in the definition of strong small system.

2. Small systems convergence.

2.1. DEFINITION. Let $\{\mathcal{N}_r\}$ be a small system on X, let $\mathcal{F}$ be the vector space of all real-valued functions on X, and let $\{f_n\}$ be a sequence in $\mathcal{F}$. We say that *f_n converges with respect to the small system $\{\mathcal{N}_r\}$ to $f \in \mathcal{F}$* if and only if, for each $\varepsilon > 0$ and for each positive integer m, there exists a positive integer $n_{\varepsilon,m}$ such that for each positive integer $n \geq n_{\varepsilon,m}$, there exists $E(n) \in \mathcal{N}_m$ such that $\{x\colon\ |f_n(x) - f(x)| \geq \varepsilon\}) \subset E(n)$.

Notice that if f_n converges with respect to a small system $\mathcal{N}_r$, then f_n converges with respect to every small system similar to $\{\mathcal{N}_r\}$.

For comparison, we include the definition for convergence in measure. Let $(X, \mathcal{S}, \mu)$ be a measure space, and let $\{f_n\}$ be a sequence of measurable functions on X. We say that f_n *converges in measure* to the measurable function f if and only if, for each $\varepsilon > 0$ and for each positive integer m, there exists

a positive integer $n_{\varepsilon,m}$ such that for each positive integer $n \geq n_{\varepsilon,m}$, we have $\mu(\{x\colon |f_n(x) - f(x)| \geq \varepsilon\}) < 1/m$ (cf. [**3**, p. 91]).

2.2. EXAMPLES. (i) Let μ be a Borel measure on a topological space X. For each r, let

$$\mathcal{N}_r = \{E\colon E \text{ is open and } \mu(E) < 1/r\}.$$

Suppose $\{f_n\}$ is a sequence of Borel functions on X and f is a Borel function on X. If f_n converges to f with respect to the small system $\{\mathcal{N}_r\}$, then f_n converges to f in measure. If f_n converges to f in measure and μ is outer regular with respect to open sets [**3**, p. 224], then f_n converges to f with respect to the small system $\{\mathcal{N}_r\}$.

(ii) Let X be the reals, and for each r, let

$$\mathcal{N}_r = \{(-\infty, -n) \cup (n, \infty)\colon n \geq r\}.$$

Convergence with respect to the small system $\{\mathcal{N}_r\}$ is precisely uniform convergence on compacta (on the real line). More generally, let X be any σ-compact space, say $X = \bigcup_{n=1}^{\infty} K_n$, where each K_n is compact and where $K_1 \subset K_2 \subset \cdots$. For each r, let

$$\mathcal{N}_r = \{X \backslash K_n\colon n \geq r\}.$$

Then convergence with respect to the small system $\{\mathcal{N}_r\}$ is the same as uniform convergence on compacta (cf. [**2**, p. 541 or **5**, p. 229]).

(iii) Let $\{\mathcal{N}_r\}$ be a small system on a set X, and suppose $\{E_n\}$ is a sequence of sets in X. It is natural to say that $\{E_n\}$ converges to E with respect to the small system $\{\mathcal{N}_r\}$ if and only if $\chi_{E_n} \to \chi_E$ with respect to the small system $\{\mathcal{N}_r\}$.

3. Null sets and equivalence of functions.

3.1. DEFINITION. Let $\{\mathcal{N}_r\}$ be a small system on X and F be a subset of X. We say that F is *null* with respect to $\{\mathcal{N}_r\}$ if and only if, for each positive integer r, there exists $E_r \in \mathcal{N}_r$ such that $F \subset E_r$.

If $\{\mathcal{N}_r\}$ and $\{M_r\}$ are similar small systems on X, it is easy to see that the null sets for $\{\mathcal{N}_r\}$ are precisely the null sets for $\{M_r\}$. It is also not difficult to verify that the null sets for $\{\mathcal{N}_r\}$ form a σ-ideal of subsets of X.

3.2. DEFINITION. Let $\{\mathcal{N}_r\}$ be a small system on X. We say that $f, g \in \mathcal{F}$ are *equivalent* (with respect to $\{\mathcal{N}_r\}$) if and only if $\{x\colon f(x) \neq g(x)\}$ is a null set with respect to $\{\mathcal{N}_r\}$.

3.3. DEFINITION. Let $\{\mathcal{N}_r\}$ be a small system on X. For each $f \in \mathcal{F}$, let $[f]$ denote the class of all $g \in \mathcal{F}$ such that g is equivalent to f, and let $\mathcal{M}$ denote the vector space of these equivalence classes. It is easy to see that if a sequence f_n converges to both f and g (with respect to $\{\mathcal{N}_r\}$), then f and g are equivalent. Moreover, if f_n converges to f and if f is equivalent to g and if, for each positive integer n, f_n is equivalent to g_n, then g_n converges to g. We may therefore say that a sequence $[f_n]$ of elements of $\mathcal{M}$ converges to $[f] \in \mathcal{M}$ with respect to the small system $\{\mathcal{N}_r\}$ if and only if the sequence f_n converges to f with respect to $\{\mathcal{N}_r\}$.

Because convergence and equivalence of functions are preserved under similarity of small systems, the space $\mathcal{M}$ and convergence in $\mathcal{M}$ are preserved under similarity of small systems.

4. Metrizability.

4.1. DEFINITION. Let $\{\mathcal{N}_r\}$ be a small system on X, let A be a subset of X, and let i be a positive integer. We say that A *adheres* to $\mathcal{N}_i$ if and only if there exists $E \in \mathcal{N}_i$ such that $A \backslash E$ is a null set with respect to $\{\mathcal{N}_r\}$.

If C adheres to $\mathcal{N}_i$ and $B \subset C$, then B clearly adheres to $\mathcal{N}_i$ also. Moreover, if a set A adheres to $\mathcal{N}_{i+1}$, then there exists $E \in \mathcal{N}_i$ such that $A \subset E$. In particular, if A adheres to $\mathcal{N}_{i+1}$, then A adheres to $\mathcal{N}_i$ also. Finally, a set A adheres to each $\mathcal{N}_r$ if and only if A is a null set.

4.2. LEMMA. *Let $\{\mathcal{N}_r\}$ be a small system on X. If $f \in \mathcal{F}$ and f is not equivalent to the zero function, then there exists a positive integer i such that $\{x\colon |f(x)| \geq 2^{-i}\}$ does not adhere to $\mathcal{N}_i$. If $g \in [f]$, then $\{x\colon |g(x)| \geq 2^{-i}\}$ also does not adhere to $\mathcal{N}_i$.*

PROOF. Suppose otherwise that for each positive integer i,

$$A_i = \{x\colon |f(x)| \geq 2^{-i}\}$$

adheres to N_i. If i and j are positive integers such that $i < j$, then $A_i \subset A_j$, so that A_i adheres to $\mathcal{N}_j$. It follows that A_i adheres to each $\mathcal{N}_r$, so that A_i is a null set. Hence, $\{x\colon f(x) \neq 0\} = \bigcup_{i=1}^{\infty} A_i$ is a null set, which violates the assumption that $f \notin [0]$. The second assertion follows from the fact that $\{x\colon f(x) \neq g(x)\}$ is a null set and its union with any null set is a null set.

4.3. LEMMA. *Let $\{\mathcal{N}_r\}$ be a strong small system on X. If $f \in \mathcal{F}$ is not equivalent to 0, let $r(f)$ be the smallest positive integer r such that $\{x\colon |f(x)| \geq 2^{-r}\}$ does not adhere to $\mathcal{N}_r$, and let $\rho(f) = 2^{-r(f)}$. If $f \in \mathcal{F}$ is equivalent to 0, let $r(f) = \infty$ and $\rho(f) = 0$. Then:*

(i) *For each $f, g \in \mathcal{F}$, if $\rho(f) = \rho(g)$, then $\rho(f+g) \leq \rho(f) + \rho(g)$.*

(ii) *For each finite set $\{f_1, \ldots, f_n\} \subset \mathcal{F}$, if the values $\{\rho(f_1), \ldots, \rho(f_n)\}$ are all distinct, then $\rho(\sum_{i=1}^n f_i)/2 \leq \max\{\rho(f_1), \ldots, \rho(f_n)\}$.*

(iii) *For each finite set $\{f_1, \ldots, f_n\} \subset \mathcal{F}$, we have $\rho(\sum_{i=1}^n f_i)/2 \leq \sum_{i=1}^n \rho(f_i)$.*

(iv) *For each $f \in \mathcal{F}$, $\rho(-f) = \rho(f)$.*

PROOF. In order to prove (i), suppose $\rho(f) = \rho(g)$. If $\rho(f) = 0$, then f and g and hence, $f+g$, are equivalent to 0, so that $\rho(f+g) = 0$. If $\rho(f) \geq \frac{1}{4}$, then we are done since $\rho(f+g) \leq \frac{1}{2}$. Hence, we may suppose that $0 < \rho(f) \leq \frac{1}{8}$, so that $r(f) \geq 3$. Let $r = r(f) = r(g)$. Then the sets $\{x\colon |f(x)| \geq 2^{-(r-1)}\}$ and $\{x\colon |g(x)| \geq 2^{-(r-1)}\}$ adhere to $\mathcal{N}_{r-1}$. It follows that their union adheres to $\mathcal{N}_{r-2}$. Hence, $\{x\colon |f(x)+g(x)| \geq 2^{-(r-2)}\}$ adheres to $\mathcal{N}_{r-2}$ since

$$\{x\colon |f(x)+g(x)| \geq 2^{-(r-2)}\} \subset \{x\colon |f(x)| \geq 2^{-(r-1)}\} \cup \{x\colon |g(x)| \geq 2^{-(r-1)}\}.$$

Thus, $r(f+g) \geq r-1$, so that $\rho(f+g) \leq 2^{-(r-1)} = 2^{-r} + 2^{-r} = \rho(f) + \rho(g)$.

To prove (ii), we may suppose that $r(f_1) < r(f_2) < \cdots < r(f_n)$. The result is clear if $\rho(f_1) \geq \frac{1}{4}$, so we may assume that $r(f_1) \geq 3$. Let $p = r(f_1) - 2$. Then $p + i < r(f_i)$ for all $i = 1, \ldots, n$, so that $\{x\colon |f_i(x)| \geq 2^{-(p+i)}\}$ adheres to $\mathcal{N}_{p+i}$ for each $i = 1, \ldots, n$. Hence, $\bigcup_{i=1}^n \{x\colon |f_i(x)| \geq 2^{-(p+i)}\}$ adheres to $\mathcal{N}_p$ (indeed, there exists $E \in \mathcal{N}_p$ such that $\bigcup_{i=1}^n \{x\colon |f_i(x)| \geq 2^{-(p+i)}\} \subset E$). Because

$$\left\{x\colon \left|\sum_{i=1}^n f_i(x)\right| \geq 2^{-p}\right\} \subset \bigcup_{i=1}^n \{x\colon |f_i(x)| \geq 2^{-(p+i)}\},$$

we see that $\{x\colon |\sum_{i=1}^n f_i(x)| \geq 2^{-p}\}$ adheres to $\mathcal{N}_p$. Hence,

$$r\left(\sum_{i=1}^n f_i\right) \geq p+1,$$

so that

$$\rho\left(\sum_{i=1}^n f_i\right) \leq 2^{-(p+1)} = 2\cdot 2^{-(p+2)} = 2\cdot 2^{-r(f_1)} = 2\rho(f_1).$$

Hence, $\rho(\sum_{i=1}^n f_i)/2 \leq \rho(f_1) = \max\{\rho(f_1),\dots,\rho(f_n)\}$.

Notice that statement (iii) holds for $n=1$. Suppose that it holds for $n=r$, and suppose $\{f_1,\dots,f_{r+1}\}$ is a collection of $r+1$ functions in F. If the numbers $\rho(f_1),\dots,\rho(f_{r+1})$ are distinct, then the statement follows from (ii). If the numbers $\rho(f_1),\dots,\rho(f_{r+1})$ are not distinct, then we may suppose that $\rho(f_r)=\rho(f_{r+1})$. Then $\rho(f_r+f_{r+1}) \leq \rho(f_r)+\rho(f_{r+1})$ from (i). By the inductive hypothesis,

$$\begin{aligned}\rho(f_1+\cdots+f_{r-1}+(f_r+f_{r+1}))/2 &\leq \rho(f_1)+\cdots+\rho(f_{r-1})+\rho(f_r+f_{r+1})\\ &\leq \rho(f_1)+\cdots+\rho(f_{r-1})+\rho(f_r)+\rho(f_{r+1}).\end{aligned}$$

That is, $\rho(\sum_{i=1}^{r+1} f_i)/2 \leq \sum_{i=1}^{r+1}\rho(f_i)$ for the family $\{f_1,\dots,f_{r+1}\}$.

Statement (iv) is obvious.

4.4. THEOREM. *Let $\{\mathcal{N}_r\}$ be a strong small system on X, and let $\rho\colon \mathcal{F}\to[0,\frac{1}{2}]$ be the function given by Lemma 4.3. Define a function $d\colon \mathcal{M}\times\mathcal{M}\to R$ by*

$$d([f],[g]) = \inf\sum_{i=1}^{n-1}\rho(f_i-f_{i+1}),$$

where the infimum is taken over all finite sets $\{f_1,\dots,f_n\}\subset\mathcal{F}$ such that $f_1=f$ and $f_n=g$. Then

(i) *for each $f,g\in\mathcal{F}$, $\rho(f-g)/2 \leq d([f],[g]) \leq \rho(f-g)$, and*

(ii) *d is a metric on $\mathcal{M}$.*

PROOF. If $g\in[f]$ and $h\in\mathcal{F}$, then $g-h$ is equivalent to $f-h$, so that $\rho(g-h)=\rho(f-h)$. It follows that d is well defined. By statement (ii) of Lemma 4.3, we have $\rho(f-g)/2 \leq d([f],[g])$. By letting $f_1=f$ and $f_2=g$, we see that $d([f],[g]) \leq \rho(f-g)$.

Because $\rho(f-g)\geq 0$ for all $f,g\in\mathcal{F}$, we have $d([f],[g])\geq 0$ for all $[f],[g]\in\mathcal{M}$ by statement (i). Again by statement (i),

$$\begin{aligned} d([f],[g])=0 \quad &\text{if and only if} \quad \rho(f-g)=0,\\ &\text{if and only if} \quad f-g \text{ is equivalent to } 0,\\ &\text{if and only if} \quad [f]=[g].\end{aligned}$$

The assertion $d([f],[g]) = d([g],[f])$ follows from statement (iv) of Lemma 4.3. More precisely, given $\varepsilon>0$, there exists a finite set $\{f_1,\dots,f_n\}\subset\mathcal{F}$ such that $f_1=f$, $f_n=g$ and

$$d([f],[g])+\varepsilon > \sum_{i=1}^{n-1}\rho(f_i-f_{i+1}).$$

For each $i = 1, \ldots, n$, let $g_i = f_{n+1-i}$. Then $\{g_1, \ldots, g_n\}$ is a finite set in $\mathcal{F}$ such that $g_1 = g$ and $g_n = f$. Hence,

$$\begin{aligned} d([g],[f]) &\le \sum_{i=1}^{n-1} \rho(g_i - g_{i+1}) = \sum_{i=1}^{n-1} \rho(f_{i+1} - f_i) \\ &= \sum_{i=1}^{n-1} \rho(f_i - f_{i+1}) < d([f],[g]) + \varepsilon. \end{aligned}$$

Because $\varepsilon > 0$ is arbitrary, we have $d([g],[f]) \le d([f],[g])$. By interchanging the roles of f and g, we get $d([f],[g]) \le d([g],[f])$, so that $d([f],[g]) = d([g],[f])$.

In order to prove that $d([f],[g]) \le d([f],[g]) + d([g],[h])$, suppose $\varepsilon > 0$ and choose finite subsets $\{f_1, \ldots, f_n\}$ and $\{g_1, \ldots, g_m\}$ in $\mathcal{F}$ such that $f_1 = f$, $f_n = g$, $g_1 = g$, and $g_m = h$ and such that

$$d([f],[g]) + \varepsilon/2 > \sum_{i=1}^{n-1} \rho(f_i - f_{i+1})$$

and

$$d([g],[h]) + \varepsilon/2 > \sum_{j=1}^{m-1} \rho(g_j - g_{j+1}).$$

Then $\{f_1, \ldots, f_n, g_1, \ldots, g_m\}$ is a finite set in $\mathcal{F}$ such that $f_1 = f$ and $g_m = h$. Hence,

$$\begin{aligned} d([f],[h]) &\le \sum_{i=1}^{n-1} \rho(f_i - f_{i+1}) + \rho(f_n - g_1) + \sum_{j=1}^{m-1} \rho(g_j - g_{j+1}) \\ &< d([f],[g]) + \varepsilon/2 + 0 + d([g],[h]) + \varepsilon/2. \end{aligned}$$

Because $\varepsilon > 0$ is arbitrary, we have $d([f],[h]) \le d([f],[g]) + d([g],[h])$.

4.5. THEOREM. *Let $\{\mathcal{N}_r\}$ be a strong small system and let d be the corresponding metric on $\mathcal{M}$ given by Theorem 4.4. A sequence $\{f_n\}$ in $\mathcal{F}$ converges to $f \in \mathcal{F}$ with respect to $\{\mathcal{N}_r\}$ if and only if* $\lim d([f_n],[f]) = 0$.

PROOF. Suppose f_n converges to f with respect to $\{\mathcal{N}_r\}$ and suppose $0 < \varepsilon < 1/2$. Let m be that positive integer such that $\varepsilon \le 2^{-m} < 2\varepsilon$. Then $2^{-m-1} < \varepsilon$ and there exists $n^* = n_{\varepsilon,m}$ such that $n \ge n^*$ implies there exists $E(n) \in \mathcal{N}_m$ such that $\{x\colon |f_n(x) - f(x)| \ge \varepsilon\} \subset E(n)$. If $n \ge n^*$, then

$$A_n = \{x\colon |f_n(x) - f(x)| \ge 2^{-m}\} \subset \{x\colon |f_n(x) - f(x)| \ge \varepsilon\} \subset E(n),$$

in which case A_n adheres to $\mathcal{N}_m$, so that $r(f_n - f) \ge m + 1$ and

$$d([f_n],[f]) \le \rho(f_n - f) \le 2^{-(m+1)} < \varepsilon.$$

Suppose $\lim d([f_n],[f]) = 0$, $\varepsilon > 0$ and m is a positive integer. Let p be a positive integer such that $2^{-p} \le \varepsilon$ and such that $p \ge m + 1$. There exists a positive integer

n^* such that

$$n \geq n^* \quad \text{implies} \quad d([f_n],[f]) < 2^{-(p+1)}.$$

Because $\rho(f_n - f)/2 \leq d([f_n],[f]) < 2^{-(p+1)}$, we have $\rho(f_n - f) < 2^{-p}$ so that $r(f_n - f) > p$ if $n \geq n^*$. Hence, if $n \geq n^*$, then

$$A_n = \{x\colon |f_n(x) - f(x)| \geq 2^{-p}\}$$

adheres to $\mathcal{N}_p$ and thus to $\mathcal{N}_{m+1}$, so that there exists $E(n) \in \mathcal{N}_m$ such that $A_n \subset E(n)$. Thus $n \geq n^*$ implies

$$\{x\colon |f_n(x) - f(x)| \geq \varepsilon\} \subset A_n \subset E(n) \in \mathcal{N}_m.$$

Therefore, f_n converges to f with respect to $\{\mathcal{N}_r\}$.

4.6. THEOREM. *Let $\{\mathcal{N}_r\}$ be a strong small system and let d be the metric on $\mathcal{M}$ given by Theorem 4.4. Then the metric space $(\mathcal{M}, d)$ is complete.*

PROOF. Let $\{[f_n]\}$ be a fundamental sequence in $(\mathcal{M}, d)$. Then there exists an increasing sequence of positive integers $\{n_i\}$ such that $d([f_{n_{i+1}}],[f_{n_i}]) < 2^{-(i+1)}$. Because $\rho(f_{n_{i+1}} - f_{n_i}) < 2^{-i}$, we see that $r(f_{n_{i+1}} - f_{n_i}) > i$, so that

$$A_i = \{x\colon |f_{n_{i+1}}(x) - f_{n_i}(x)| \geq 2^{-i}\}$$

adheres to $\mathcal{N}_i$. We let $A = \limsup A_i$ and show that the sequence $\{f_{n_i}(x)\}$ has a limit for each $x \in X \backslash A$.

Let $x \in X \backslash A$. Since x belongs to only finitely many A_i's, there exists i_0 such that $i \geq i_0$ implies $x \in X \backslash A_i$, and this implies $|f_{n_{i+1}}(x) - f_{n_i}(x)| < 2^{-i}$. Hence, for each $x \in X \backslash A$, the series $f_{n_1}(x) + \sum_{i=1}^{\infty}(f_{n_{i+1}}(x) - f_{n_i}(x))$ is absolutely convergent to some finite number $f(x)$. We put $f(x) = 0$ for $x \in A$. If $j \geq 2$ and $x \in X \backslash \bigcup_{i=j}^{\infty} A_i$, then

$$|f(x) - f_{n_j}(x)| \leq \sum_{i=j}^{\infty} |f_{n_{i+1}}(x) - f_{n_i}(x)|.$$

Because $\{x\colon |f(x) - f_{n_j}(x)| \geq 2^{-(j-1)}\} \subset \bigcup_{i=j}^{\infty} A_i$ and because $\bigcup_{i=j}^{\infty} A_i$ adheres to $\mathcal{N}_{j-1}$, we see that $r(f - f_{n_j}) \geq j$ so that $\rho(f - f_{n_j}) \leq 2^{-j}$ and $d([f],[f_{n_j}]) \leq 2^{-j}$. We now show that the fundamental sequence $\{[f_n]\}$ converges to $[f]$ in $(\mathcal{M}, d)$. Suppose $\varepsilon > 0$. Choose a positive integer k such that $2^{-(k-1)} < \varepsilon$. Then there exists n^* such that if $n, m \geq n^*$, then $d([f_m],[f_n]) < 2^{-k}$. Choose a positive integer j such that $j \geq \max\{2, k\}$ and such that $n_j \geq n^*$. Then $d([f],[f_{n_j}]) \leq 2^{-j} \leq 2^{-k}$, and $n \geq n^*$ implies $d([f_{n_j}],[f_n]) < 2^{-k}$. Hence, $n \geq n^*$ implies $d([f],[f_n]) < 2^{-k} + 2^{-k} = 2^{-(k-1)} < \varepsilon$. Thus $(\mathcal{M}, d)$ is complete.

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