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THE VALIDITY OF BEURLING THEOREMS IN POLYDISCS

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ABSTRACT. This paper gives necessary and sufficient conditions for an invariant subspace $\mathscr{M}$ of $H^2(T^2)$ to be of the form $qH^2(T^2)$ (q inner) in terms of double commutativity of the shifts. Recent results in [**8**] follow directly from our work. Relation to the work in [**1**] is also discussed.

In [**5**], Rudin gave an example of a shift-invariant subspace of $H^2(T^2)$ which is not of the form $qH^2(T^2)$, where q is an inner function. In addition, [**5**] gives an example of an "outer function" f for which $\mathscr{M}_f$, the smallest invariant subspace generated by f is not equal to $H^2(T^2)$. Motivated from the prediction theory for random fields, Soltani [**8**] gave complicated necessary and sufficient conditions for an "outer function" in the sense of [**5**] to satisfy $\mathscr{M}_f = H^2(T^2)$. Our purpose here is to use the Theorem of Halmos for two commuting isometries due to Slocinski [**7**] to obtain necessary and sufficient conditions on the invariant subspace $\mathscr{M}$ of $H^2(T^2)$ to be of the form $qH^2(T^2)$ (q inner). As a consequence of this result we obtain necessary and sufficient conditions on $\mathscr{M}_f$ for an outer function f, to be $H^2(T^2)$. We use results of [**3**] to relate our conditions to those in [**8**]. As a by-product of our main theorem we show that all invariant subspaces unitarily equivalent to $qH^2(T^2)$ are exactly of the same form. The last result relates to some recent work of Agrawal, Clark, and Douglas [**1**], where the problem of unitary equivalence of invariant subspaces of $H^2(T^n)$ was solved under some sufficient conditions. Their case does not include subspaces of the type considered here. We thank the referee and the editor for bringing the work in [**1**] to our attention. We begin with some notation.

Let $\mathbf{Z}$ be the set of integers. We denote by m, n etc. the elements of $\mathbf{Z}$. Let U be the open unit disc and T the boundary of U in the complex plane $\mathbf{C}$. Let $\mathbf{Z}^2, \mathbf{C}^2, U^2$ and T^2 be the respective cartesian products and σ_2 the normalized Lebesgue measure on T^2. For $p > 0$, we denote by $L^p(T^2, \sigma_2)$ the usual Lebesgue space of the equivalence class of p-integrable functions and

$$H^p(U^2) = \left\{ f : f : U^2 \to \mathbf{C} \text{ analytic and } \sup_{0 \le r \le 1} \int_T |f_r(\mathbf{t})|^p \, d\sigma_2 < \infty \right\}.$$

Here $f_r(\mathbf{t}) = f(z)$ with $z = r\mathbf{t}$. Let $\mathbf{z} = (z_1, z_2) = (r_1 e^{i\theta_1}, r_2 e^{i\theta_2})$ and $\mathbf{t} = (e^{i\theta_1}, e^{i\theta_2})$, then $P(\mathbf{z}, \mathbf{t}) = P_{r_1}(\theta_1 - \theta_2)$. $P_{r_2}(\theta_1 - \theta_2)$ is called Poisson kernel with

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$P_r(\theta) = (1-r^2)/(1-2r\cos\theta+r^2)$. It is known that for $f \in H^p(U^2)$, $\lim_{r\to 1} f_r(\mathbf{t}) = f^*(\mathbf{t})$ exists and is in $L^p(T^2,\sigma_2)$. For $f \in L^p(T^2,\sigma_2)$, let $f^e(z) = \int_{T^2} P(\mathbf{z},\mathbf{t})f(t)\,d\sigma_2$, then $f^e \in H^p(U^2)$. In case $p = 2$, $f^e \in H^2(U^2)$ if $f \in L^2(T^2,\sigma_2)$ and $f(\mathbf{t}) = \sum_{m=0}\sum_{n=0} a_{mn}t_1^m t_2^n$ and $f^e = \sum_{m=0}\sum_{n=0} a_{mn}z_1^m z_2^n$. Conversely, every $f \in H^2(U^2)$ has this form and $f^*(\mathbf{t}) = \sum_{m=0}\sum_{n=0} a_{mn}t_1^m t_2^n$. For further information see [**5**].

In order to characterize invariant subspaces of the form qH^2 in terms of the section of the shifts on it, we note that subspaces of the form qH^2 can be represented as

$$qH^2 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\bigoplus V_1^m V_2^n(M) \tag{1}$$

where M equals the span of q in $H^2(T^2)$ and V_i is the multiplication by t_i on $H^2(T^2)$ with $\mathbf{t} = (t_1,t_2) \in T^2$. As V_1 commutes with V_2 (in short, $V_1 \sim V_2$), we get from (1) and Theorem 1 (i)$\Rightarrow$(ii) of [**7**] that V_1 and V_2 are doubly commuting (i.e. $V_1 \sim V_2, V_1 \sim V_2^*$). In fact, we have

2. THEOREM. *An invariant subspace $\mathscr{M} \neq \{0\}$ of $H^2(T^2)$ is of the form $q\cdot H^2$ with q inner function if and only if V_1 and V_2 are doubly commuting on $\mathscr{M}$.*

PROOF. Necessity was proved above. To prove the sufficiency we get, in view of Theorem 1 ((ii)$\Rightarrow$(iv)) [**7**],

$$\mathscr{M} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\bigoplus V_1^m V_2^n(R_1^\perp \cap R_2^\perp), \tag{3}$$

where $R_i^\perp = \mathscr{M} \ominus V_i\mathscr{M}$. From Theorem 1 (i)$\Leftrightarrow$(iv) and Corollary 1 of [**7**, p. 256] we get that $R_1^\perp \cap R_2^\perp = \{0\}$ implies $\mathscr{M} = \{0\}$ giving contradiction. Hence $R_1^\perp \cap R_2^\perp \neq \{0\}$. We shall now prove that $R_1^\perp \cap R_2^\perp$ is one-dimensional. The above argument shows that dimension of $R_1^\perp \cap R_2^\perp$ is at least one. It remains to prove that it is not greater than one. Let $q_1, q_2 \in R_1^\perp \cap R_2^\perp$, then $\int_{T^2} t_1^m t_2^n q_1 q_2 d\sigma_2 = 0$ for $m,n > 0$. As $V_2(R_1^\perp) \subseteq (R_1^\perp)$ [**7**, Theorem 1 (iii)], we get $\int_{T^2} t_2^n q_1 \overline{t_1^m q_2}\, d\sigma_2 = 0$ for all $n > 0$ and $m > 0$. Since $t_1^m = t_1^{-m}$, by the symmetry of the problem, we get for $(m,n) \neq (0,0)$ $\int t_1^m t_2^n q_1 \bar{q}_2\, d\sigma_2 = 0$. Since $q_1\bar{q}_2 \in L^1(T^2,\sigma_2)$ we get $q_1\bar{q}_2 = c$ a.e. σ_2. Suppose now that $q_1, q_2 \in R_1^\perp \cap R_2^\perp$ $q_1 \neq 0$, $q_2 \neq 0$ and $q_1 \perp q_2$. This implies that $|q_1|^2 = c_1 \neq 0$ a.e. σ_2, $|q_2|^2 = c_2 \neq 0$ a.e. σ_2 but $q_1\bar{q}_2 = 0$. The last statement follows from the fact $q_1 \perp q_2$ and a.e. σ_2, $q_1\bar{q}_2 = c$ giving $c = 0$ (integrating with respect to σ_2). This is impossible. Hence $R_1^\perp \cap R_2^\perp$ is one dimensional. Also q generating $R_1^\perp \cap R_2^\perp$ is an inner function. Assume $|q| = 1$ a.e. choosing q of norm 1. Now (3) gives the result.

In a recent paper [**1**], Agrawal, Clark and Douglas study the question of unitary equivalence of invariant subspaces of H^2 of polydiscs. Under the assumption of full range of an invariant subspace $\mathscr{M}$ (for definition see [**1**, (2), p. 5]), they show that all invariant subspaces $\mathscr{N}$ of H^2 unitarily equivalent to $\mathscr{M}$ are of the form $\varphi\mathscr{M}$ (φ an inner function). Unfortunately, subspaces of the form qH^2 do not have full range [**1**, Remark 3, p. 6] unless q is constant. We use our result to study all invariant subspaces unitarily equivalent to qH^2. We consider the case of subspaces of $H^2(T^2)$.

If $\mathscr{M}$ is an invariant subspace unitarily equivalent to qH^2 for some inner function, then in view of Theorem 2, we get V_1 and V_2 are doubly commuting on $\mathscr{M}$. Hence $\mathscr{M} = q'H^2$ for some inner function q' by Theorem 2 with $|q'(\mathbf{t})| = 1$ a.e. σ_2. We thus have the following.

4. THEOREM. (a) *The class of all invariant subspaces of the form qH^2, q inner function with $|q(\mathbf{t})| = 1$ a.e. σ_2 are unitarily equivalent.*

(b) *Any invariant subspace $\mathscr{M}$ of $H^2(T^2)$ unitarily equivalent to $\mathscr{N} = qH^2$ of the above form is of the same form.*

REMARK. Note that in (b) $\mathscr{M} = q'/q\mathscr{N}$. Comparing this with Lemma 1 of [**1**] we get that the function $\varphi = q'/q$ with q' and q relatively prime in the sense of [**1**, p. 4].

In [**6**], Rudin gives an example of the subspace $\mathscr{M} = q'/q\mathscr{N}$ with q', q inner but not relatively prime. We should also note that there exists subspaces of the form qH^2 which do not satisfy analytic condition used in §3 of [**1**].

The example of Rudin [**5**] mentioned in the introduction shows that there exists $f \in H^2$ such that

$$\mathscr{M}_f = \overline{sp}\{V_1^m V_2^n f; m, n \geq 0\}$$

is not of the form qH^2. The following Corollary gives conditions on $\mathscr{M}_f$ for this to hold.

5. COROLLARY. *$\mathscr{M}_f = qH^2(T^2)$ if and only if V_1 and V_2 are doubly commuting on $\mathscr{M}_f$.*

Following Helson [**2**], we say that a function g is H-outer if $\mathscr{M}_g = H^2(T^2)$.

6. COROLLARY. *A function $f \in H^2(T^2)$ has the property $f = q \cdot g$ with q inner and g H-outer if and only if V_1 and V_2 doubly commute on $\mathscr{M}_f$.*

PROOF. By Corollary 5, only if part follows as $\mathscr{M}_f = qH^2(T^2)$. To prove the converse, we note that by Corollary 5, $\mathscr{M}_f = qH^2(T^2)$ given $f = q \cdot g$, $g \in H^2(T^2)$. Hence $\mathscr{M}_f = q \cdot \mathscr{M}_g$ giving g is H-outer.

In [**5**, p. 73], a function $f \in H^2(U^2)$ is called outer (we call it R-outer) if $\log|f(z)| = \int_{T^2} P(z,t) \log|f^*|\, d\sigma_2$. Given a function $f \in H^2(T^2)$, we denote by $f^e \in H^2(U^2)$ the function given by $\int_{T^2} P(z,t) f(\mathbf{t})\, d\sigma_2$. In this case we note that $(f^e)^* = f$. It is already known [**5**, Theorem 4.4.6] that f is H-outer then f^e is R-outer. From this we get in view of Corollary 5 the following.

7. COROLLARY. *Let g be H-outer then g^e is R-outer and V_1 and V_2 doubly commute on $\mathscr{M}_g$.*

We now prove the converse of Corollary 7. Assume now that V_1, V_2 doubly commute on $\mathscr{M}_f$ and f^e is R-outer then by Corollaries 6 and 7, the definition of f^e, g^e and the fact that $|q| = 1$ we get $f^e = pg^e$ with $|p| = 1$. Thus we get that the slice function $f^e_w(\lambda) = p_w(\lambda) g^e_w(\lambda)$. Using Lemma 4.4.4(a) of [**5**] and the uniqueness of outer function [**2**, p. 13] we get $p_w(\lambda) = 1$ for all w and λ giving $p = 1$ i.e., $f = g$.

Combining this with Corollary 7 gives

8. COROLLARY. *Let $f \in H^2(T^2)$ then $\mathscr{M}_f = H^2(T^2)$ if and only if f^e is R-outer and V_1 and V_2 doubly commute.*

In view of Theorem 4.2 of [**3**], we get that Corollary 7 includes Beurling Theorem proved in [**8**, Theorem 1.5]. Now using essentially classical techniques [**4**, **3**] one can derive associated results in prediction theory given in [**8**].

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