THE VALIDITY OF BEURLING THEOREMS IN POLYDISCS

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ABSTRACT. This paper gives necessary and sufficient conditions for an invariant subspace \( \mathcal{M} \) of \( H^2(T^2) \) to be of the form \( qH^2(T^2) \) (\( q \) inner) in terms of double commutativity of the shifts. Recent results in [8] follow directly from our work. Relation to the work in [1] is also discussed.

In [5], Rudin gave an example of a shift-invariant subspace of \( H^2(T^2) \) which is not of the form \( qH^2(T^2) \), where \( q \) is an inner function. In addition, [5] gives an example of an "outer function" \( f \) for which \( \mathcal{M}_f \), the smallest invariant subspace generated by \( f \) is not equal to \( H^2(T^2) \). Motivated from the prediction theory for random fields, Soltani [8] gave complicated necessary and sufficient conditions for an "outer function" in the sense of [5] to satisfy \( \mathcal{M}_f = H^2(T^2) \). Our purpose here is to use the Theorem of Halmos for two commuting isometries due to Slocinski [7] to obtain necessary and sufficient conditions on the invariant subspace \( \mathcal{M} \) of \( H^2(T^2) \) to be of the form \( qH^2(T^2) \) (\( q \) inner). As a consequence of this result we obtain necessary and sufficient conditions on \( \mathcal{M}_f \) for an outer function \( f \), to be \( H^2(T^2) \). We use results of [3] to relate our conditions to those in [8]. As a by-product of our main theorem we show that all invariant subspaces unitarily equivalent to \( qH^2(T^2) \) are exactly of the same form. The last result relates to some recent work of Agrawal, Clark, and Douglas [1], where the problem of unitary equivalence of invariant subspaces of \( H^2(T^n) \) was solved under some sufficient conditions. Their case does not include subspaces of the type considered here. We thank the referee and the editor for bringing the work in [1] to our attention. We begin with some notation.

Let \( \mathbb{Z} \) be the set of integers. We denote by \( m, n \) etc. the elements of \( \mathbb{Z} \). Let \( U \) be the open unit disc and \( T \) the boundary of \( U \) in the complex plane \( \mathbb{C} \). Let \( \mathbb{Z}^2, \mathbb{C}^2, U^2 \) and \( T^2 \) be the respective cartesian products and \( \sigma_2 \) the normalized Lebesgue measure on \( T^2 \). For \( p > 0 \), we denote by \( L^p(T^2, \sigma_2) \) the usual Lebesgue space of the equivalence class of \( p \)-integrable functions and

\[ H^p(U^2) = \left\{ f : f : U^2 \to \mathbb{C} \text{ analytic and } \sup_{0 \leq r \leq 1} \int_T |f_r(t)|^p \, d\sigma_2 < \infty \right\}. \]

Here \( f_r(t) = f(z) \) with \( z = rt \). Let \( z = (z_1, z_2) = (r_1e^{i\theta_1}, r_2e^{i\theta_2}) \) and \( t = (e^{i\theta_1}, e^{i\theta_2}) \), then \( P(z, t) = P_{r_1}(\theta_1 - \theta_2) \). \( P_{r_2}(\theta_1 - \theta_2) \) is called Poisson kernel with
Pr(θ) = (1 − r²)/(1 − 2r cos θ + r²). It is known that for f ∈ Hp(U²), \lim_{r→1} f_r(t) = f^*(t) exists and is in \text{L}^p(T², σ_2). For f ∈ \text{L}^p(T², σ_2), let f^*(z) = \int_{T²} P(z, t)f(t) dσ_2, then f^* ∈ Hp(U²). In case p = 2, f^* ∈ H²(U²) if f ∈ \text{L}^2(T², σ_2) and f(t) = \sum_{m=0}^{∞} \sum_{n=0}^{∞} a_{mn}t²_mz²_n and f^* = \sum_{m=0}^{∞} \sum_{n=0}^{∞} a_{mn}t²_mz²_n. Conversely, every f ∈ H²(U²) has this form and f^*(t) = \sum_{m=0}^{∞} \sum_{n=0}^{∞} a_{mn}t²_mz²_n. For further information see [5].

In order to characterize invariant subspaces of the form qH² in terms of the section of the shifts on it, we note that subspaces of the form qH² can be represented as

\begin{equation}
qH² = \bigoplus_{m=0}^{∞} \bigoplus_{n=0}^{∞} V_1^mV_2^n(M)
\end{equation}

where M equals the span of q in H²(T²) and Vi is the multiplication by ti on H²(T²) with t = (t₁, t₂) ∈ T². As V₁ commutes with V₂ (in short, V₁ ∼ V₂), we get from (1) and Theorem 1 (i)⇒(ii) of [7] that V₁ and V₂ are doubly commuting (i.e. V₁ ∼ V₂, V₁ ∼ V₂). In fact, we have

2. THEOREM. An invariant subspace \mathcal{M} ≠ {0} of H²(T²) is of the form q·H²

with q inner function if and only if V₁ and V₂ are doubly commuting on \mathcal{M}.

PROOF. Necessity was proved above. To prove the sufficiency we get, in view of Theorem 1 ((ii)⇒(iv)) [7],

\begin{equation}
\mathcal{M} = \bigoplus_{m=0}^{∞} \bigoplus_{n=0}^{∞} V_1^mV_2^n(R_1^+ ∩ R_2^+),
\end{equation}

where R_1^+ = \mathcal{M} ∩ V₁. From Theorem 1 (i)⇒(iv) and Corollary 1 of [7, p. 256] we get that R_1^+ ∩ R_2^+ = {0} implies \mathcal{M} = {0} giving contradiction. Hence R_1^+ ∩ R_2^+ ≠ {0}. We shall now prove that R_1^+ ∩ R_2^+ is one-dimensional. The above argument shows that dimension of R_1^+ ∩ R_2^+ is at least one. It remains to prove that it is not greater than one. Let q₁, q₂ ∈ R_1^+ ∩ R_2^+, then \int_{T²} t²_mq₁q₂ dσ₂ = 0 for m, n > 0. As V₂(R_1^+) ⊆ (R₂^+) [7, Theorem 1 (iii)], we get \int_{T²} t²_mq₁q₂ dσ₂ = 0 for all n > 0 and m > 0. Since t²_m = t⁻², by the symmetry of the problem, we get for (m, n) ≠ (0, 0) \int t²_mq₁q₂ dσ₂ = 0. Since q₁q₂ ∈ L¹(T², σ₂) we get q₁q₂ = c a.e. σ₂. Suppose now that q₁, q₂ ∈ R₁^+ ∩ R₂^+ q₁ = 0, q₂ ≠ 0 and q₁ ⊥ q₂. This implies that |q₁|^² = c₁ = 0 a.e. σ₂, |q₂|^² = c₂ = 0 a.e. σ₂ but q₁q₂ = 0. The last statement follows from the fact q₁ ⊥ q₂ and a.e. σ₂, q₁q₂ = c giving c = 0 (integrating with respect to σ₂). This is impossible. Hence R_1^+ ∩ R_2^+ is one dimensional. Also q generating R_1^+ ∩ R_2^+ is an inner function. Assume |q| = 1 a.e. choosing q of norm 1. Now (3) gives the result.

In a recent paper [1], Agrawal, Clark and Douglas study the question of unitary equivalence of invariant subspaces of H² of polydiscs. Under the assumption of full range of an invariant subspace \mathcal{M} (for definition see [1, (2), p. 5]), they show that all invariant subspaces \mathcal{N} of H² unitarily equivalent to \mathcal{M} are of the form \varphi\mathcal{M} (\varphi an inner function). Unfortunately, subspaces of the form qH² do not have full range [1, Remark 3, p. 6] unless q is constant. We use our result to study all invariant subspaces unitarily equivalent to qH². We consider the case of subspaces of H²(T²).
If $\mathcal{M}$ is an invariant subspace unitarily equivalent to $qH^2$ for some inner function, then in view of Theorem 2, we get $V_1$ and $V_2$ are doubly commuting on $\mathcal{M}$. Hence $\mathcal{M} = q'H^2$ for some inner function $q'$ by Theorem 2 with $|q'(t)| = 1$ a.e. $\sigma_2$. We thus have the following.

4. **Theorem.** (a) The class of all invariant subspaces of the form $qH^2$, $q$ inner function with $|q(t)| = 1$ a.e. $\sigma_2$ are unitarily equivalent.

(b) Any invariant subspace $\mathcal{M}$ of $H^2(T^2)$ unitarily equivalent to $\mathcal{N} = qH^2$ of the above form is of the same form.

**Remark.** Note that in (b) $\mathcal{M} = q'/q\mathcal{N}$. Comparing this with Lemma 1 of [1] we get that the function $\varphi = q'/q$ with $q'$ and $q$ relatively prime in the sense of [1, p. 4].

In [6], Rudin gives an example of the subspace $\mathcal{M} = q'/q\mathcal{N}$ with $q'$, $q$ inner but not relatively prime. We should also note that there exists subspaces of the form $qH^2$ which do not satisfy analytic condition used in §3 of [1].

The example of Rudin [5] mentioned in the introduction shows that there exists $f \in H^2$ such that
\[
\mathcal{M}_f = \text{sp}\{V_1^nV_2^m f; m, n \geq 0\}
\]
is not of the form $qH^2$. The following Corollary gives conditions on $\mathcal{M}_f$ for this to hold.

5. **Corollary.** $\mathcal{M}_f = qH^2(T^2)$ if and only if $V_1$ and $V_2$ are doubly commuting on $\mathcal{M}_f$.

Following Helson [2], we say that a function $g$ is $H$-outer if $\mathcal{M}_g = H^2(T^2)$.

6. **Corollary.** A function $f \in H^2(T^2)$ has the property $f = q \cdot g$ with $q$ inner and $g$ $H$-outer if and only if $V_1$ and $V_2$ doubly commute on $\mathcal{M}_f$.

**Proof.** By Corollary 5, only if part follows as $\mathcal{M}_f = qH^2(T^2)$. To prove the converse, we note that by Corollary 5, $\mathcal{M}_f = qH^2(T^2)$ given $f = q \cdot g$, $g \in H^2(T^2)$. Hence $\mathcal{M}_f = q \cdot \mathcal{M}_g$ giving $g$ is $H$-outer.

In [5, p. 73], a function $f \in H^2(U^2)$ is called outer (we call it $R$-outer) if
\[
\log|f(z)| = \int_{T^2} P(z,t) \log|f^*| \, d\sigma_2.
\]
Given a function $f \in H^2(T^2)$, we denote by $f^e \in H^2(U^2)$ the function given by $\int_{T^2} P(z,t)f(t) \, d\sigma_2$. In this case we note that $(f^e)^* = f$. It is already known [5, Theorem 4.4.6] that $f$ is $H$-outer then $f^e$ is $R$-outer. From this we get in view of Corollary 5 the following.

7. **Corollary.** Let $g$ be $H$-outer then $g^e$ is $R$-outer and $V_1$ and $V_2$ doubly commute on $\mathcal{M}_g$.

We now prove the converse of Corollary 7. Assume now that $V_1, V_2$ doubly commute on $\mathcal{M}_f$ and $f^e$ is $R$-outer then by Corollaries 6 and 7, the definition of $f^e, g^e$ and the fact that $|q| = 1$ we get $f^e = pg^e$ with $|p| = 1$. Thus we get that the slice function $f^e_w(\lambda) = p_w(\lambda)g^e_w(\lambda)$. Using Lemma 4.4.4(a) of [5] and the uniqueness of outer function [2, p. 13] we get $p_w(\lambda) = 1$ for all $w$ and $\lambda$ giving $p = 1$ i.e., $f = g$. 

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Combining this with Corollary 7 gives

8. **COROLLARY.** Let \( f \in H^2(T^2) \) then \( \mathcal{M}_f = H^2(T^2) \) if and only if \( f^* \) is R-outer and \( V_1 \) and \( V_2 \) doubly commute.

In view of Theorem 4.2 of [3], we get that Corollary 7 includes Beurling Theorem proved in [8, Theorem 1.5]. Now using essentially classical techniques [4, 3] one can derive associated results in prediction theory given in [8].

**REFERENCES**


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