WEIGHTED NORM INEQUALITIES FOR BOCHNER-RIESZ SPHERICAL SUMMATION MULTIPLIERS

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ABSTRACT. Sufficient conditions to be satisfied by nonnegative weight functions \( u(|x|) \) are given in order that the Bochner-Riesz spherical summation multiplier operators restricted to radial functions of \( \mathbb{R}^n \) be bounded on \( L^p(\mathbb{R}^n; u(|x|)dx) \). For a certain class of weights these conditions are also necessary.

1. Introduction. Let \( \hat{f}(\xi) \) denote the Fourier transform of \( f \) and let \( B = \{ \xi : |\xi| \leq 1 \} \) be the unit ball in \( \mathbb{R}^n \). For \( \lambda \geq 0 \) the Bochner-Riesz spherical summation multiplier operator \( T_\lambda = T_{n,\lambda} \) is defined by

\[
(T_\lambda f)(x) = c_{n,\lambda} \int_{\mathbb{R}^n} f(y) J_{n/2+\lambda}(|x-y|) \frac{dy}{|x-y|^{n/2+\lambda}}
\]

where \( J_\nu \) is the Bessel function of the first kind of order \( \nu \).

\( L^p(\mathbb{R}^n; u(x)dx) \), abbreviated \( L^p(\mathbb{R}^n) \) when \( u(x) = 1 \), denotes the weighted Lebesgue space of measurable functions \( f \) on \( \mathbb{R}^n \) for which \( \int_{\mathbb{R}^n} |f(x)|^p u(x) dx < \infty \). The boundedness of \( T_\lambda \) on \( L^p \) spaces has been widely studied because of its intimate connection with the inversion of the Fourier transform by Bochner-Riesz means.

If \( \lambda \) exceeds the critical index \( \lambda_c = (n-1)/2 \), (1.1) shows that \( T_\lambda \) is a convolution operator with kernel in \( L^1(\mathbb{R}^n) \) so \( T_\lambda \) is bounded on \( L^p(\mathbb{R}^n) \) for all \( p, 1 \leq p \leq \infty \); moreover, since in this case \( T_\lambda f \) is dominated by a multiple of the Hardy-Littlewood maximal function of \( f \) [15, Theorem 2, p. 62], a result of B. Muckenhoupt [13] shows that \( T_\lambda \) is bounded on \( L^p(\mathbb{R}^n; u(x)dx) \), \( 1 < p < \infty \), if \( u \) satisfies

\[
(A_p(\mathbb{R}^n)) \quad \left( \int_Q \omega(x) dx \right)^{1/p} \left( \int_Q \omega(x)^{-p'/p} dx \right)^{1/p'} \leq C \int_Q dx
\]

for all cubes \( Q \subset \mathbb{R}^n \) with sides parallel to the coordinate axis.

When \( n = 1 \), \( T_0 \) reduces essentially to the Hilbert transform and hence \( T_0 \) is bounded on \( L^p(\mathbb{R}^1; \omega(x)dx) \) if (and only if) \( \omega \) satisfies \( A_2(\mathbb{R}^1) \), (see [10]) while for \( n > 1 \) C. Fefferman [6] has shown that \( T_0 \) is bounded on \( L^p(\mathbb{R}^n) \) only if \( p = 2 \).

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For $0 < \lambda \leq \lambda_c$, the boundedness properties of $T_\lambda$ are not yet fully understood. Herz [8] has shown that $T_\lambda$ is unbounded if $p \leq p_0$ or $p \geq p'_0$ where $p_0 = p_0(\lambda, n) = 2n/(n + 1 + 2\lambda)$ and it has been conjectured that $T_\lambda$ is bounded for $p_0 < p < p'_0$. The conjecture has been verified for $n = 2$ by L. Carleson and P. Sjölin [3] but for $n \geq 3$ it has been verified only for $\lambda > (n - 1)/2(n + 1)$, see [5, 7, 17].

The restriction of $T_\lambda$ to radial functions in $\mathbb{R}^n$ is bounded on $L^p$ for $p_0 < p < p'_0$, $0 \leq \lambda \leq \lambda_c$, see [8, 19, 4]. S. Chanillo and B. Muckenhoupt [4] proved weak type $(p_0, p_0)$ estimates for the restricted operator when $0 < \lambda \leq \lambda_c$ but as C. Kenig and P. Tomas [11] have shown, these do not extend to the case $\lambda = 0$ when $n > 1$.

I. Hirschman, Jr. [9] obtained the boundedness of $T_\lambda$, $0 \leq \lambda < \lambda_c$, on the power weighted space $L^2(\mathbb{R}^n; |x|^\alpha \, dx)$ for $|\alpha| < 1 + 2\lambda$.

The main result of this paper gives sufficient conditions on the weight function $\omega$ which ensure that $T_\lambda$ restricted to radial functions is bounded on $L^p(\mathbb{R}^n; \omega(|x|) \, dx)$. For a certain class of weights of the form $\omega(r) = r^\alpha(1 + r)^{\beta - \alpha}$ these conditions are also seen to be necessary. These results generalize those of [1] where the case $\lambda = 0$ was treated.

**THEOREM 1.** Let $1 < p < \infty$, $1/p + 1/p' = 1$, $\lambda > 0$ and suppose $\omega(|x|)$ is nonnegative on $\mathbb{R}^n$ and satisfies
\[
\left( \int_a^b \frac{\omega(r)r^{n-1}}{(1 + r)(n-1-2\lambda)p/2} \, dr \right)^{1/p} \left( \int_a^b \frac{\omega(r)^{-p'/p}r^{n-1}}{(1 + r)(n-1-2\lambda)p'/2} \, dr \right)^{1/p'} \leq K \int_a^b \frac{r^{n-1}}{(1 + r)^{n-1-2\lambda}} \, dr
\]
for some constant $K = K_{n, \lambda, p, \omega}$ and all $0 \leq \alpha < b < \infty$. Then there is a constant $C = C_{n, \lambda, p, K}$ such that
\[
\int_{\mathbb{R}^n} |(T_\lambda f)(x)|^p \omega(|x|) \, dx \leq C \int_{\mathbb{R}^n} |f(x)|^p \omega(|x|) \, dx
\]
for all radial functions $f$ in $L^p(\mathbb{R}^n; \omega(|x|) \, dx)$.

**COROLLARY.** If $-n < \alpha < n(p - 1)$ then (1.3) holds with $\omega(r) = r^\alpha(1 + r)^{\beta - \alpha}$ if and only if
\[
-n + (n - 1 - 2\lambda)p/2 < \beta < -n + (n + 1 + 2\lambda)p/2.
\]

The corollary may be proved as follows. It is easily verified that (1.2) holds with $\omega(r) = r^\alpha(1 + r)^{\beta - \alpha}$ if $-n < \alpha < n(p - 1)$ and $\beta$ satisfies (1.4) so the sufficiency follows directly from the theorem. Since $|(T_\lambda \chi_B)(x)| \sim c_{n, \lambda} |x|^{-(n+1+2\lambda)/2}$ as $|x| \to \infty$, and $\chi_B \in L^p(\mathbb{R}^n; \omega(|x|) \, dx)$ for $\alpha > -n$, (1.3) with $f = \chi_B$ shows the necessity of the upper bound for $\beta$ in (1.4). On the other hand, $\chi_B \in L^{p'}(\mathbb{R}^n; \omega(|x|)^{-p'/p} \, dx)$ for $\alpha < n(p - 1)$ so a standard duality argument then shows the necessity of the lower bound for $\beta$ in (1.4).

Note that with $\alpha = \beta = 0$ the corollary recovers the $L^p(\mathbb{R}^n)$ boundedness result for $p_0 < p < p'_0$ cited above while with $p = 2$ and $\alpha = \beta$ (1.4) coincides with the range obtained by Hirschman for the unrestricted operator.
Let $\Omega(n, \lambda, p)$ denote the class of $\omega$ which satisfy (1.2). The following properties are easily verified:

\begin{align*}
(1.5) \quad n_1 < n_2 & \Rightarrow \Omega(n_1, \lambda, 2) \subset \Omega(n_2, \lambda, 2), \\
(1.6) \quad \lambda_1 < \lambda_2 & \Rightarrow \Omega(n, \lambda_1, p) \subset \Omega(n, \lambda_2, p), \\
(1.7) \quad p_1 < p_2 & \Rightarrow \Omega(n, \lambda, p_1) \subset \Omega(n, \lambda, p_2).
\end{align*}

The examples provided by the corollary show that the analogues of (1.5) for $p \neq 2$ and of (1.7) for $\lambda \neq \lambda_c$ do not hold. Note also that if $\omega(r) = r^\alpha$ then $\omega \in \Omega(n, \lambda, p)$ if and only if $\omega(|x|)$ satisfies $A_p(R^n)$.

Theorem 1 may be generalized somewhat as follows. For radial functions $f(x) = f_0(|x|)$ on $R^n$, $(T_n, \lambda f)(x) = (R_n, \lambda f_0)(|x|)$ is also radial and if $P_k(x)$ is a harmonic homogeneous polynomial of degree $k$ in $R^n$, then

$$(T_n, \lambda f P_k)(x) = (-1)^k (R_{n+2k}, \lambda f_0)(|x|) P_k(x),$$

see [16, Chapter IV, §§2, 3]. This with Theorem 1 yields Theorem 2.

**Theorem 2.** Let $1 < p < \infty$, $1/p + 1/p' = 1$, $\lambda > 0$, $k \geq 0$ an integer and suppose $\omega(|x|)$ is nonnegative on $R^n$ and satisfies

$$\int_a^b \frac{\omega(r)}{(1 + r)^{(n+2k-1-2\lambda)p'/2}} dr \leq K \int_a^b \frac{r^{n+2k-1}}{(1 + r)^{(n+2k-1-2\lambda)p'/2}} dr$$

for some constant $K = K_{n, \lambda, p, \omega}$ and all $0 \leq a < b < \infty$. Then there is a constant $C = C_{n, \lambda, p, k, K}$ such that

$$\int_{R^n} |(T_n, \lambda f P_k)(x)|^p \omega(|x|) dx \leq C \int_{R^n} |f(x) P_k(x)|^p \omega(|x|) dx$$

for all radial functions $f$ and all harmonic homogeneous polynomials $P_k$ of degree $k$ in $R^n$ with $f P_k$ in $L^p(R^n; \omega(|x|) dx)$.

**2. Proof of Theorem 1.** We follow the usual practice that $C$ will denote an absolute constant, not necessarily the same from line to line. The proof requires the following lemmas. Define the operators $P = P_{n, \lambda}$, $Q = Q_{n, \lambda}$, and $M$ for $s > 0$ by

\begin{align*}
(P\phi)(s) &= (1 + s)^{-(n+1+2\lambda)/2} \int_0^s (1 + r)^{-(n-1)/2} \phi(r) r^{n-1} dr, \\
(Q\phi)(s) &= (1 + s)^{-(n-1)/2} \int_s^\infty (1 + r)^{-(n+1+2\lambda)/2} \phi(r) r^{n-1} dr, \\
(M\phi)(s) &= \sup_{0 < h < s} \frac{1}{2h} \int_{s-h}^{s+h} \phi(r) dr.
\end{align*}

**Lemma 1.** If $\lambda > 0$ and $f(x) = f_0(|x|)$ is a simple radial function on $R^n$ then there is a constant $C = C_{n, \lambda}$ such that

$$|(T_\lambda f)(x)| \leq C \{ (P + Q)(|x|) + (M f_0 \chi_{[|x|/2, 2|x|]})(|x|) \}.$$

**Proof.** Set $|x| = s$, $|y| = r$ and consider separately the cases $s \leq 4$ and $s > 4$. 

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If \( s \leq 4 \), (1.1) and the estimates [18, pp. 49, 199]

\[
|J_\nu(t)| \leq C_\nu t^\nu, \quad |J_\nu(t)| \leq C_\nu (1 + t)^{-1/2}, \quad t > 0,
\]

show that \(|(T_\lambda f)(x)|\) does not exceed

\[
C_{n,\lambda} \left\{ \int_0^8 \left| f_0(r) \right| r^{n-1} \, dr + \int_8^\infty (1 + r)^{-(n+1+2\lambda)/2} \left| f_0(r) \right| r^{n-1} \, dr \right\}
\]

since \( s \leq 4 \) and \( r \geq 8 \) implies \(|x - y| \geq r - s \geq r/2 \geq (1 + r)/4\). Thus \(|(T_\lambda f)(x)| \leq C_{n,\lambda} \{(|P + Q||f_0||f_0|)(s)\} \) for \( s \leq 4 \).

Now let \( s > 4 \) and set \( f = f_1 + f_2 \), where \( f_1(y) = (f_0\chi_{(0,2)}(|y|)) \). Then (1.1) and (2.2) show that

\[
|(T_\lambda f_1)(x)| \leq C_{n,\lambda} \int_0^2 (1 + s)^{-(n+1+2\lambda)/2} |f_0(r)| r^{n-1} \, dr \leq C_{n,\lambda} (P|f_0|)(s)
\]
since in this case \(|x - y| \geq s - r \geq s/2 \geq (1 + s)/4\). The estimates (3.3), (3.4) and (3.5) of [4, p. 701] show that \(|(T_\lambda f_2)(x)|\) does not exceed a constant multiple of

\[
(1 + r)^{-\alpha/2} \int_2^{s/2} (1 + r)^{-(n-1)/2} |f_0(r)| r^{n-1} \, dr
\]

\[
+ (Mf_0\chi_{[s/2,2s]})(s)
\]

and this is bounded by

\[
C_{n,\lambda} \{(|P + Q||f_0||f_0|)(s) + (Mf_0\chi_{[s/2,2s]})(s)\}.
\]

Thus, \(|(T_\lambda f)(x)| \leq |(T_\lambda f_1)(x)| + |(T_\lambda f_2)(x)|\) shows that (2.1) holds for \( s > 4 \) also.

This proves Lemma 1.

**Lemma 2.** Let \( 1 < p < \infty, \ 1/p + 1/p' = 1 \) and suppose \( \omega \geq 0 \) on \((0, \infty)\). The following statements are equivalent:

(a) There is a constant \( C = C_{n,\lambda,p,\omega} \) such that

\[
\int_0^\infty |(P\phi)(s)|^p \omega(s) s^{n-1} \, ds \leq C \int_0^\infty |\phi(r)|^p \omega(r) r^{n-1} \, dr.
\]

(b) There is a constant \( C = C_{n,\lambda,p,\omega} \) such that

\[
\int_0^\infty |(Q\psi)(s)|^p \omega(s)^{-p'/p} s^{n-1} \, ds \leq C \int_0^\infty |\psi(r)|^p \omega(r)^{-p'/p} r^{n-1} \, dr.
\]

(c) There is a constant \( C = C_{n,\lambda,p,\omega} \) such that for all \( a > 0 \)

\[
\left( \int_a^\infty \frac{\omega(r)^{n-1}}{(1 + r)^{(n+1+2\lambda)/2}} \, dr \right)^{1/p} \left( \int_0^a \frac{\omega(r)^{-p'/p} r^{n-1}}{(1 + r)^{(n-1)/2}} \, dr \right)^{1/p'} \leq C.
\]

**Proof.** Since \( \int_0^\infty \psi(s)|P\phi(s)| s^{n-1} \, ds = \int_0^\infty \phi(r)(Q\psi)(r) r^{n-1} \, dr \) for \( \phi, \psi \geq 0 \), a standard duality argument shows that (a) and (b) are equivalent. Theorem 1 of [12] yields the equivalence of (a) and (c).
**Lemma 3.** Let $1 < p < \infty$, $a_0 \geq 0$, and $V(r) \geq 0$. If for some $\varepsilon > 0$ there is a constant $C = C_{p,a_0,\varepsilon,V}$ such that for all $a > a_0$

$$
\left( \int_a^\infty \left( \frac{a}{r} \right)^{\varepsilon} V(r) \frac{dr}{r} \right)^{1/p} \left( \int_0^a V(r)^{-p'/p} dr \right)^{1/p'} \leq C
$$

then there is a constant $C = C_{p,a_0,\varepsilon,V}$ such that for all $a > a_0$

$$
\left( \int_a^\infty \frac{V(r) dr}{r^p} \right)^{1/p} \left( \int_0^a V(r)^{-p'/p} dr \right)^{1/p'} \leq C.
$$

**Proof.** This is proved in [2, Lemma 2] for the case $a_0 = 0$; the general case is entirely similar and is therefore omitted.

Returning to the proof of Theorem 1, observe first that (1.2) shows $\omega$ and $\omega^{-p'/p}$ are locally integrable on $[0, \infty)$. Hence the set of simple functions is dense in both $L^p(R^n; \omega(|x|) dx)$ and $L^{p'}(R^n; \omega(|x|)^{-p'/p} dx)$ and therefore by Lemma 1 it suffices to show that

(2.3) $$
\int_0^\infty |(Pf_0)(s)|^p \omega(s) s^{n-1} ds,
$$

(2.4) $$
\int_0^\infty |(Qf_0)(s)|^p \omega(s) s^{n-1} ds
$$

and

(2.5) $$
\int_0^\infty |(Mf_0\chi_{[s/2,2s]})(s)|^p \omega(s) s^{n-1} ds
$$

are each bounded by a constant multiple of $\int_0^\infty |f_0(r)|^p \omega(r) r^{n-1} dr$.

Consider (2.3) first. For $t > a > 0$, (1.2) implies there is a constant $C = C_{n,\lambda,p,K}$ such that

$$
\left( \int_a^t \frac{\omega(r)r^{n-1}}{(1+r)^{(n-1-2\lambda)p/2}} dr \right) \left( \int_0^a \frac{\omega(r)^{-p'/p} r^{n-1}}{(1+r)^{(n-1-2\lambda)p'/2}} dr \right)^{p/p'} \leq K^p \left( \int_0^t \frac{r^{n-1}}{(1+r)^{(n-1-2\lambda)p}} dr \right)^p \leq C(1+t)^{(2\lambda+1)p}.
$$

Multiplying this by $(1+t)^{-(2\lambda+1)p-2}$, integrating the result over $a < t < \infty$ and using Fubini's theorem on the left shows that

$$
\left( \int_a^\infty \frac{(1+a)}{(1+r)^{(n+1+2\lambda)p/2}} dr \right) \left( \int_0^a \frac{\omega(r)^{-p'/p} r^{n-1}}{(1+r)^{(n-1-2\lambda)p'/2}} dr \right)^{p/p'}
$$

is bounded by a constant $C$. Thus, Lemma 3 shows that

(2.6) $$
\left( \int_a^\infty \frac{\omega(r)r^{n-1}}{(1+r)^{(n+1+2\lambda)p/2}} dr \right)^{1/p} \left( \int_0^a \frac{\omega(r)^{-p'/p} r^{n-1}}{(1+r)^{(n-1-2\lambda)p'/2}} dr \right)^{1/p'}
$$

is bounded by $C$ for all $a \geq 1$. If $a < 1$, the first factor of (2.6) is bounded by a constant times the sum of

$$
\left( \frac{1}{(1+a)^{(1+2\lambda)p}} \int_a^1 \frac{\omega(r)r^{n-1}}{(1+r)^{(n-1-2\lambda)p/2}} dr \right)^{1/p}
$$
and

\[
\left( \int_1^\infty \frac{\omega(r) r^{n-1}}{(1 + r)(n+1+2\lambda)p/2} dr \right)^{1/p}.
\]

Thus, for \( a < 1 \), (2.6) is bounded by the sum of two terms, the first of which is bounded because of (1.2) while the second is bounded because of (2.6) with \( a = 1 \). Thus for \( a > 0 \) we have

\[
\left( \int_a^\infty \frac{\omega(r) r^{n-1}}{(1 + r)(n+1+2\lambda)p/2} dr \right)^{1/p} \left( \int_0^a \frac{\omega(r)^{-p'/p} r^{n-1}}{(1 + r)(n-1-2\lambda)p'/2} dr \right)^{1/p'} \leq C
\]

so parts (a) and (c) of Lemma 2 show that (2.3) has the required bound.

That (2.4) has the required bound now follows easily from the duality expressed by parts (a) and (b) of Lemma 2 since (1.2) is self-dual in the sense that \( \omega \) satisfies (1.2) for a given \( p \) if and only if (1.2) is satisfied with \( \omega \) and \( p \) replaced by \( \omega^{-p'/p} \) and \( p' \).

Finally, to show that (2.5) has the required bound, observe first that (1.2) shows there is a constant \( C = C_{n,\lambda,K} \) independent of the integer \( m \) such that

\[
\left( \int_a^b \omega(r) dr \right)^{1/p} \left( \int_a^b \omega(r)^{-p'/p} dr \right)^{1/p'} \leq C(b - a)
\]

whenever \( 2^m - 1 \leq a < b \leq 2^{m+2} \). Thus Muckenhoupt’s well-known result [13] for the Hardy-Littlewood maximal function shows that

\[
\int_{2^{m-1}}^{2^{m+2}} |(M\phi)(s)|^p \omega(s) ds \leq C \int_{2^{m-1}}^{2^{m+2}} |\phi(r)|^p \omega(r) dr
\]

for a constant \( C = C_{n,\lambda,K} \) independent of \( m \). Hence (2.5) is bounded by

\[
\sum_{m=-\infty}^{\infty} \int_{2^m}^{2^{m+1}} |(M\chi_{[2^m,2^{m+2}]})(s)|^p \omega(s) s^{n-1} ds
\]

\[
\leq C \sum_{m=-\infty}^{\infty} 2^{(m+1)(n-1)} \int_{2^{m-1}}^{2^{m+2}} |\phi(r)|^p \omega(r) dr
\]

\[
\leq C \int_0^\infty |\phi(r)|^p \omega(r) r^{n-1} dr
\]

as required. This completes the proof of Theorem 1.

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