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INTEGRAL INEQUALITIES OF HARDY AND POINCARÉ TYPE

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ABSTRACT. The Poincaré inequality $\|u\|_p \leq C\|\nabla u\|_p$ in a bounded domain holds, for instance, for compactly supported functions, for functions with mean value zero and for harmonic functions vanishing at a point. We show that it can be improved to $\|u\|_p \leq C\|\delta^\beta \nabla u\|_p$, where δ is the distance to the boundary, and the positive exponent β depends on the smoothness of the boundary.

1. Introduction. In this note we improve standard versions of Poincaré's inequality by applying Hardy's inequality for bounded domains Ω in $\mathbf{R}^n$. Our work was stimulated by a recent paper of Ziemer in which he showed [**Z**, §3] that if the boundary of Ω is locally the graph of a continuous function, then for every linear second-order elliptic equation there is a constant C such that

$$\|u\|_p \leq C\|\nabla u\|_p \tag{1.1}$$

for every solution u normalized by $u(x_0) = 0$. Here $\|u\|_p$ is the $L^p(\Omega)$ norm of u ($1 \leq p < \infty$) and ∇ denotes the gradient. We show that if the boundary of Ω is locally the graph of a Hölder continuous function of exponent α then the right-hand side of (1.1) can be replaced by $C\|\delta^\alpha \nabla u\|_p$, where δ denotes the distance to the boundary. A similar improvement of Poincaré's inequality holds for many other function classes; see §2.

We denote the space of functions u with norm $\|u\|_p + \|\delta^\alpha \nabla u\|_p < \infty$ by $W^{1,p}(\Omega, \alpha)$, or just $W^{1,p}(\Omega)$ when $\alpha = 0$, while $W^{1,p}_{\text{loc}}(\Omega)$ denotes the space of functions that lie in $W^{1,p}(\omega)$ for every $\omega \Subset \Omega$. Our abstract version of Poincaré's inequality is the following

THEOREM. *Let Ω be a bounded domain in $\mathbf{R}^n$ whose boundary is locally the graph of a Hölder continuous function of exponent α, where $0 \leq \alpha \leq 1$, and suppose $1 \leq p < \infty$. Let H be a cone in $W^{1,p}_{\text{loc}}(\Omega)$ such that the closure of $H \cap W^{1,p}(\Omega, \alpha)$ in $W^{1,p}(\Omega, \alpha)$ contains no nonzero constant function. Then there is a constant C such that*

$$\|u\|_p \leq C\|\delta^\alpha \nabla u\|_p \tag{1.2}$$

for every function u in H, where δ denotes the distance to the boundary of Ω.

It is part of the Theorem that finiteness of the right-hand side implies finiteness of the left-hand side. When $\alpha = 0$ we understand the boundary of Ω to be locally

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the graph of simply a continuous function, and we recapture (1.1) with no gain. When $\alpha = 1$ the boundary of Ω is locally the graph of a Lipschitz function—this is equivalent [**G**, Theorem 1.2.2.2] to the uniform cone condition for Ω, and holds, for instance, for every convex domain—and we attain the maximal gain of a full power of δ in (1.2).

We give the proof of the Theorem in §3. In §2 we obtain improvements of various standard formulations of Poincaré's inequality by specifying the cone H. In §4 we show that the Theorem is sharp and also indicate some extensions.

2. Examples.

EXAMPLE 2.1. Suppose $\int_\Omega \phi \, dV \neq 0$ for a function ϕ in $L^q(\Omega)$, where $p^{-1} + q^{-1} = 1$. The set $H := \{u \in L^p(\Omega) \cap W^{1,p}_{\text{loc}}(\Omega) : \int_\Omega u\phi \, dV = 0\}$ satisfies the assumptions of the Theorem. In particular, the Theorem applies to the class of functions with mean value zero (take ϕ to be identically one).

EXAMPLE 2.2. The set $H := \{u \in W^{1,p}_{\text{loc}}(\Omega) : u$ vanishes on a set of measure at least $\gamma\}$ satisfies the assumptions of the Theorem, where γ is a fixed positive number.

EXAMPLE 2.3. Suppose $p > n$. In this case, $W^{1,p}_{\text{loc}}(\Omega)$ embeds in the space of continuous functions by Sobolev's lemma [**Ad**, Theorem 5.4], so if x_0 is a fixed point in Ω the Theorem applies with $H := \{u \in W^{1,p}_{\text{loc}}(\Omega) : u(x_0) = 0\}$.

EXAMPLE 2.4. Let P be a linear partial differential operator on Ω with smooth coefficients. Suppose that P is *hypoelliptic*; that is, every solution u of the equation $Pu = f$ is smooth on every open set where f is. The Theorem applies with $H := \{u \in W^{1,p}_{\text{loc}}(\Omega) : Pu = 0$ and $u(x_0) = 0\}$. A nonzero constant cannot lie in the closure of $H \cap W^{1,p}(\Omega, \alpha)$ because, by the hypoellipticity of P and the closed graph theorem, convergence in this space implies convergence in $C^\infty(\Omega)$.

The class of hypoelliptic operators contains, besides the elliptic operators, parabolic operators such as the heat operator, and also certain operators that arise in the theory of the $\bar{\partial}$-Neumann problem in several complex variables. See Chapter III and §1 of Chapter XV of [**T**] for a discussion and further references.

EXAMPLE 2.5. Consider a linear second-order equation

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j} + b_i u\right) + \sum_{i=1}^{n} c_i \frac{\partial u}{\partial x_i} + du = 0 \tag{2.1}$$

in divergence form, where the coefficients a_{ij}, b_i, c_i, and d are only assumed to be locally bounded in Ω, and suppose (2.1) is locally strictly elliptic in the sense that for every compact set $K \subset \Omega$ there is a positive λ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \quad \text{for all } x \in K \text{ and } \xi \in \mathbf{R}^n.$$

The Theorem applies with $H := \{u \in W^{1,2}_{\text{loc}}(\Omega) : u$ is a weak solution of (2.1) and $u(x_0) = 0\}$. Indeed by Theorem 8.24 of [**GT**] there is some $\gamma > 0$ for which

$$\|u\|_{C^\gamma(\omega)} \leq C\|u\|_{L^2(\Omega)}, \tag{2.2}$$

where $x_0 \in \omega \Subset \Omega$. Hence $H \cap W^{1,2}(\Omega, \alpha)$ contains no nonzero constant in its closure. The special case $\alpha = 0$ is essentially the result of §3 of [**Z**].

In the previous two examples we considered solutions of *linear* partial differential equations. Since the set H is not required to be a subspace, but only a cone, certain homogeneous *nonlinear* partial differential equations are also within the scope of the Theorem. However, for general nonlinear equations the constant C must be allowed to depend on the norm of u: see §2 of [**Z**] where the case of second-order nonlinear elliptic equations is considered.

EXAMPLE 2.6. Suppose $H := C_0^\infty(\Omega)$ is the space of smooth compactly supported functions in Ω. The estimate $\|\delta^{-\alpha}u\|_p \le \|\nabla u\|_p$ for u in $C_0^\infty(\Omega)$ follows from [**Kf**, Theorem 8.4] and the classical Poincaré inequality, while some related inequalities with the weight entirely on the left-hand side are discussed in the recent papers [**An** and **Le**]. Our Theorem implies that part, but not all, of the weight can be moved to the right-hand side.

PROPOSITION. *Let Ω be a bounded domain in $\mathbf{R}^n$ whose boundary is locally the graph of a Hölder continuous function of exponent α, where $0 < \alpha \le 1$, and suppose $1 < p < \infty$. Then for all u in $C_0^\infty(\Omega)$,*

$$\|\delta^{-1/p}u\|_p \le C\|\delta^{\alpha-1/p}\nabla u\|_p \quad \text{if } 1/p \le \alpha < 1;$$
$$\|\delta^{-\varepsilon-1/p}u\|_p \le C\|\delta^{1-\varepsilon-1/p}\nabla u\|_p \quad \text{if } \alpha = 1 \text{ and } 0 < \varepsilon \le 1 - 1/p.$$

PROOF. By applying the classical one-variable Hardy inequality [**HLP**, Theorem 330] along a direction transverse to the boundary, one obtains [**Kf**, Theorem 8.4]

$$\|\delta^{-\beta}u\|_p \le C(\|\delta^{\alpha-\beta}\nabla u\|_p + \|\delta^{\alpha-\beta}u\|_p), \tag{2.4}$$

where $\beta = 1/p$ if $1/p \le \alpha < 1$ and $\beta = \varepsilon + 1/p$ (for any $\varepsilon > 0$) if $\alpha = 1$. Hence the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega, \alpha - \beta)$ contains no nonzero constant u, since the left-hand side of (2.4) is infinite for such a u. The Theorem implies $\|u\|_p \le C\|\delta^{\alpha-\beta}\nabla u\|_p$ for u in $C_0^\infty(\Omega)$, and combining this with (2.4) gives the result of the Proposition.

The weaker inequality $\|u\|_p \le C\|\delta^{1-\varepsilon-1/p}\nabla u\|_p$ appears in [**Kd**, Theorem 12.8] for Ω with twice differentiable boundary. It is interesting that this estimate cannot be improved; that is, removing the negative power of δ from the left-hand side does not make it possible to increase the power of δ on the right-hand side. This is immediate from the following result, which extends Proposition 9.10 of [**Kf**] to the case $\beta = 1 - p^{-1}$.

LEMMA. *Let Ω be a bounded domain in $\mathbf{R}^n$ whose boundary is locally the graph of a Lipschitz continuous function. Then $C_0^\infty(\Omega)$ is dense in $W^{1,p}(\Omega, \beta)$ if $\beta \ge 1 - p^{-1} > 0$.*

PROOF. It suffices to find compactly supported functions approximating $u \in C^\infty(\overline{\Omega})$, since this space is dense [**Kf**, proof of Theorem 7.2] in $W^{1,p}(\Omega, \beta)$, and so it is enough to consider the case $\beta = 1 - p^{-1}$. Let $\Delta(x)$ be a regularized distance function [**St**, p. 171] for Ω. Let $\phi_n(t)$ be a smooth function on $\mathbf{R}$ that vanishes outside the interval $(4^{-n}, 2^{-n})$ and closely approximates the function $(nt \log 2)^{-1}$ in $L^p(4^{-n}, 2^{-n})$, and set $\psi_n(x) = \int_0^{\Delta(x)} \phi_n(t)\,dt$. Then $\psi_n \in C_0^\infty(\Omega)$ and $\psi_n u \to u$ in $W^{1,p}(\Omega, 1 - p^{-1})$ because $\psi_n \to 1$ in $L^p(\Omega)$ and $\int_\Omega \delta^{p-1}|\nabla\psi_n|^p = O(n^{1-p}) \to 0$; the latter follows from Fubini's theorem by first integrating locally in the direction transverse to the boundary.

3. Proof of the Theorem. The proof results from combining Hardy's inequality with a compactness argument that goes back at least to Morrey [**M**, p. 83]. The version of Hardy's inequality that we need (using different weights from those above) is a special case of Theorem 8.2 of [**Kf**]: there exists a smoothly bounded relatively compact subdomain ω of Ω such that

$$\|u\|_p \leq C(\|\delta^\alpha \nabla u\|_p + \|u\|_{L^p(\omega)}) \tag{3.1}$$

for every locally integrable function u. The statement in [**Kf**] does not include the case $\alpha = 0$ nor the case $p = 1$, and the error term in (3.1) is given there as $\|\delta^\alpha u\|_p$, but the stronger statement given here is implicit in the proof.

In view of (3.1), to prove the Theorem we need only show that $\|u\|_{L^p(\omega)} \leq C\|\delta^\alpha \nabla u\|_p$ for every function u in H. If this estimate did not hold, there would be a sequence of functions u_j in H such that

$$\|u_j\|_{L^p(\omega)} = 1 \tag{3.2}$$

and

$$\|\delta^\alpha \nabla u_j\|_p < 1/j. \tag{3.3}$$

In particular, the u_j form a bounded sequence in $W^{1,p}(\omega)$, which embeds compactly in $L^p(\omega)$ by the Rellich-Kondrashov theorem [**Ad**, Theorem 6.2]. By passing to a subsequence we may assume that the u_j converge in $L^p(\omega)$ to some limit u, and in view of (3.1) and (3.3) the convergence even takes place in $W^{1,p}(\Omega, \alpha)$. But $\|u\|_{L^p(\omega)} = 1$ by (3.2), and the gradient of u vanishes identically by (3.3). Hence u is a nonzero constant, which contradicts the hypothesis on H and proves the Theorem.

4. Further results. (1) The Theorem is sharp in the following three senses. Without the hypothesis that the boundary be locally a graph, estimate (1.1) may fail for function classes other than $C_0^\infty(\Omega)$: see p. 521 of [**CH**], [**H**], and Theorem 10 of [**AS**]. The exponent α in (1.2) cannot be increased: if Ω is the planar domain $\{(x, y)\colon 0 < x < 1,\ |y| < x^{1/\alpha}\}$, where $0 < \alpha \leq 1$, and $\beta > \alpha$, then $1 + \alpha^{-1} \leq \gamma p < 1 + \alpha^{-1} + (\beta - \alpha)p/\alpha$ implies $\|\delta^\beta \nabla z^{-\gamma}\|_p < \infty$ but $\|z^{-\gamma}\|_p = \infty$. When u is harmonic, an inequality $\|\delta \nabla u\|_p \leq C\|u\|_p$ in the reverse directions holds because of the subaveraging property of $|u|^p$: see for instance [**D**, Lemma 1].

(2) Analogous statements, with a loss of ε in the power of δ, can be proved for $p = \infty$ by replacing the Rellich-Kondrashov theorem with the Arzelà-Ascoli theorem and Hardy's inequality with the convergence of $\int_\Omega \delta^{-\beta}$ for $\beta < \alpha$. For instance, one obtains that if u has mean value zero in a bounded convex domain Ω, then $\|u\|_\infty \leq C\|\delta^\beta \nabla u\|_\infty$ for every $\beta < 1$.

(3) By the same method one can establish weighted Poincaré inequalities of the form $\|\delta^\gamma u\|_p \leq C\|\delta^\beta \nabla u\|_p$ with $0 < \gamma < \beta$. Even certain weights more general than powers are admissible (cf. [**Kf**, §12]).

(4) If the operator P in Example 2.4 satisfies the strong maximum principle and if Ω has (say) once continuously differentiable boundary (so that $\alpha = 1$), then (1.2) can be refined to $\|u\|_p \leq C\|\delta X u\|_p$, where X is any smooth vector field that is everywhere transverse to the boundary. It is easy to see (by integrating along integral curves of X) that Xu can replace ∇u in the generalized Hardy inequality (3.1). Hence if there were a sequence of functions u_j in H with $\|u_j\|_{L^p(\omega)} = 1$ and

$\|\delta X u_j\|_p < 1/j$ the u_j would be a bounded set in $L^p(\Omega)$. By the hypoellipticity a subsequence converges in $C^\infty(\Omega)$ to a solution of $Pu = 0$ with $Xu = 0$. But if u is constant along the integral curves of X then u takes its maximum in the interior and so is constant. We obtain a contradiction as in §3 since in this case a nonzero constant cannot lie in even the $L^p(\Omega)$ closure of H. This phenomenon has recently found applications in several complex variables.

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