PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 103, Number 1, May 1988

POINT SINGULARITIES AND CONFORMAL METRICS ON RIEMANN SURFACES

ROBERT C. MCOWEN

(Communicated by Irwin Kra)

ABSTRACT. Given a closed hyperbolic Riemann surface and a finite number of points, we prove the existence and uniqueness of hyperbolic conformal metrics with prescribed singularities or degeneracies at the given points.

If M is a closed Riemann surface with negative Euler characteristic $\chi(M)$, then it admits a compatible metric g with Gauss curvature $K \equiv -1$. If $p \in M$, then we can ask for a compatible metric $\hat{g}$ on $\hat{M} = M \setminus \{p\}$ with Gauss curvature $\hat{K} \equiv -1$ *and* some prescribed singularity or degeneracy at p,

$$\hat{g}/g = O(r^{2\alpha}) \quad \text{as } r = r(x) = \operatorname{dist}_g(x,p) \to 0. \tag{1}$$

Such singularities arise, for example, from maps which are locally $z \to z^m$ ($z \in \mathbf{C}$, $m \in \mathbf{Z}^+$): pushing the standard metric forward gives a singularity corresponding to $\alpha = -(m-1)/m$ and pulling back the standard metric gives a degeneracy corresponding to $\alpha = m-1$. Thus we are particularly interested in (1) with $-1 < \alpha < \infty$.

More generally, we can consider a finite number of points $p_1, \ldots, p_n \in M$ and $\alpha_1, \ldots, \alpha_n \in \mathbf{R}$ and try to find a compatible metric $\hat{g}$ on $\hat{M} = M \setminus \{p_1, \ldots, p_n\}$ with

$$\hat{g}/g = O(r_i^{2\alpha_i}) \quad \text{as } r_i = r_i(x) = \operatorname{dist}_g(x,p_i) \to 0. \tag{2}$$

Our main result is the following.

THEOREM. *Let (M,g) be a compact Riemann surface with Gauss curvature $K \equiv -1$ and $p_1, \ldots, p_n \in M$. Suppose the numbers $\alpha_1, \ldots, \alpha_n$ satisfy* (i) $-1 < \alpha_i < \infty$, *and* (ii) $\chi(M) + \sum_i \alpha_i < 0$. *Then $\hat{M} = M \setminus \{p_1, \ldots, p_n\}$ admits a unique metric $\hat{g}$ which is pointwise conformal to g, has Gauss curvature $\hat{K} \equiv -1$, and satisfies* (2). *Moreover, $\hat{g}$ has total curvature*

$$\int_M (-1)\, d\hat{A} = 2\pi \left(\chi(M) + \sum_i \alpha_i \right). \tag{3}$$

PROOF. We shall assume for notational convenience that $n = 1$ but all steps of the proof generalize immediately. We want to solve

$$\Delta u - e^{2u} = -1 \quad \text{on } \hat{M} = M \setminus \{p\} \tag{4}$$

Received by the editors November 14, 1986 and, in revised form, February 24, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 30F10; Secondary 35J60.
Key words and phrases. Conformal metrics, Riemann surfaces, point singularities.

with $u \in C^\infty(\hat{M})$ of the form $u = u_1 + v$ where u_1 is singular, behaving like $\alpha \ln r$ as $r \to 0$, and v is bounded; in fact we will have $v \in C^\beta(M)$ for some $\beta > 0$, indeed $v \in C^2(M)$ if $\alpha > 0$. Then $\hat{g} = e^{2u} g$ satisfies (1).

To define u_1, let δ be so small that $D = \{x \in M : r(x) < 2\delta\}$ is a disk and let $G(x)$ be the Green's function for D with singularity at p, so $G(x) = -(2\pi)^{-1} \ln(r(x)) + O(1)$ as $r(x) \to 0$. Choose $u_1 \in C^\infty(\hat{M})$ so that $u_1(x) = -2\pi\alpha G(x)$ for $r(x) < \delta$. Notice that

$$\int_M \Delta u_1 \, dA = -2\pi\alpha. \tag{5}$$

We now must find a bounded solution v of

$$\Delta v = He^{2v} - \Delta u_1 - 1 \quad \text{on } M \tag{6}$$

where $H(x) = \exp[2u_1(x)] = O(r^{2\alpha})$ as $r \to 0$. We can reduce this problem even further. Let $u_0 \in C^\infty(M)$ satisfy

$$\Delta u_0 = -1 - \Delta u_1 - 2\pi(\chi(M) + \alpha)$$

(notice the right-hand side has integral zero). If we let $w = v - u_0$ and $c = 2\pi(\chi(M) + \alpha)$, then we want to solve

$$\Delta w = c + he^{2w} \quad \text{on } \hat{M} \tag{7}$$

where $h = H \exp[2u_0] > 0$, $h = O(r^{2\alpha})$ as $r \to 0$, and $c < 0$ by (ii).

The form of this equation is familiar from [**2**], and the method of upper and lower solutions employed in [**2**] to solve it need only be modified slightly to allow the mild singularity of h.

By (i) we find $h \in L^p(M)$ for $1 \le p < p_0$ where $p_0 = \infty$ (if $\alpha > 0$) or $p_0 = -1/\alpha$ (if $-1 < \alpha < 0$). Thus we may define $\overline{h} = (\int h \, dA)/(\int dA)$ and solve $\Delta\varphi = h - \overline{h}$ with $\varphi \in H_2^p(M)$ and so $\varphi \in C^\beta(M)$ for some $\beta > 0$; in particular, φ is bounded. Choose a and $b > 0$ so that $\overline{h} > -c/a$ and $\exp[2a\varphi + 2b] > a$. Then $w_+ = a\varphi + b$ satisfies $\Delta w_+ < c + he^{2w_+}$. Now let $k(x) = \max(1, h(x))$ and $\mu > 0$ so that $\overline{k} = -c/\mu$. Then we can solve $\Delta\psi = \mu k + c$ with $\psi \in H_2^p(M)$ for all p, and so $\psi \in C^{1+\beta}(M)$ for all $0 < \beta < 1$; in particular, ψ is bounded. Choose λ sufficiently large that $w_- = \psi - \lambda$ satisfies $w_- < w_+$ and $\Delta w_- > c + he^{2w_-}$.

REMARK. If $\alpha < 0$ we could take $w_+ =$ large positive constant whereas if $\alpha > 0$ then we could take $w_- =$ large negative constant; however this fails if $n > 1$ and the α_i having varying sign.

We cannot apply Lemma 9.3 of [**2**] directly since we will not have $w_+ \in C^1(M)$ if $-1 < \alpha \le -1/2$. However, $w_\pm \in C^1(\hat{M})$ so we can apply the monotone iteration scheme on a sequence of closed sets M_j with $\hat{M} = \bigcup M_j$ (cf. the proof of Theorem 1 in [**1**]). We obtain a bounded solution w of (7). Since $h \in C^\infty(\hat{M}) \cap L^p(M)$ for $1 \le p < p_0$, elliptic regularity shows $w \in C^\infty(\hat{M}) \cap C^\beta(M)$ for some $\beta > 0$ (in fact $w \in C^2(M)$ if $\alpha > 0$). If we let $S_\delta(p) = \{x \in M : r(x) = \delta\}$ and ν denote the unit normal (towards p) then

$$\int_M \Delta w \, dA = \lim_{\delta \to 0} \int_{S_\delta} \frac{\partial w}{\partial \nu} \, d\sigma = 0. \tag{8}$$

Thus $u = u_1 + u_0 + w$ solves (4); moreover

$$\int_M (-1)\, d\hat{A} = \int_M (-1)e^{2u}\, dA = \int_M (-1 - \Delta u)\, dA$$

yields (3) by (5) and (8).

To verify uniqueness, suppose $\hat{g}_1$ and $\hat{g}_2$ are 2 such metrics. Since they both satisfy (1) this means that there are 2 solutions, w_1 and w_2, of (7). If $\alpha > 0$ then $w_1, w_2 \in C^2(M)$ so $w_1 = w_2$ by the maximum principle. Otherwise, let $w_0 = w_1 - w_2$ which satisfies $\Delta w_0 = h(e^{2w_1} - e^{2w_2})$ and $w_0 \in C^\infty(\hat{M}) \cap C^\beta(M)$. Although w_0 achieves its maximum and minimum on M, it cannot achieve a positive maximum or negative minimum on $\hat{M}$ by the maximum principle. So suppose w_0 achieves a positive maximum at p: for some small neighborhood U of p we have $w_0(x) > 0$ for $x \in U$ and $\partial w_0/\partial \nu \leq 0$ on ∂U where ν is the outward normal on ∂U. But, arguing as in (8), we find

$$0 < \int_U \Delta w_0\, dA = \int_{\partial U} \frac{\partial w_0}{\partial \nu}\, d\sigma \leq 0$$

a contradiction. Similarly we find that we cannot have a negative minimum at p. Thus $w_1 = w_2$ and the metric $\hat{g}$ is unique.

REMARK. Note that the metric $\hat{g}$ is incomplete on $\hat{M}$ by (i). In fact, if $\alpha \leq -1$ then it can be shown (e.g. by an argument similar to the proof of Lemma B in [**3**]) that (6) admits no bounded solution. Hence condition (i) is necessary, i.e. g admits no complete conformal metric $\hat{g}$ satisfying (1) and $\hat{K} \equiv -1$. On the other hand, by uniformization theory, it is known that $(\hat{M}, g)$ *does* admit a complete conformal metric $\hat{g}$ with $\hat{K} \equiv -1$. In fact, if we replace g by a conformal metric which is Euclidean near p, i.e. $g = dx^2 + dy^2$ and $K \equiv 0$ near p, then it is possible to solve $\Delta u - e^{2u} = K$ on $\hat{M}$ with $u \geq -\ln r - \ln|\ln r| - C$ as $r = (x^2 + y^2)^{1/2} \to 0$ where C is a constant. Thus (1) is replaced by $\hat{g}/g = O(r^{-2}|\ln r|^{-2})$ as $r \to 0$.

ACKNOWLEDGMENTS. The author wishes to thank K. Uhlenbeck and W. Goldman for bringing this problem to his attention, and UCSD where the work was conducted.

REFERENCES

1. P. Aviles and R. McOwen, *Conformal deformations of complete manifolds with negative curvature*, J. Differential Geom. **21** (1985), 269–281.
2. J. Kazdan and F. Warner, *Curvature functions for compact 2-manifolds*, Ann. of Math. (2) **99** (1974), 14–47.
3. R. McOwen, *On the equation $\Delta u + Ke^{2u} = f$ and prescribed negative curvature in $\mathbf{R}^2$*, J. Math. Anal. Appl. **103** (1984), 365–370.

DEPARTMENT OF MATHEMATICS, NORTHEASTERN UNIVERSITY, BOSTON, MASSACHUSETTS 02115