POINT SINGULARITIES AND CONFORMAL METRICS ON RIEMANN SURFACES

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(Communicated by Irwin Kra)

ABSTRACT. Given a closed hyperbolic Riemann surface and a finite number of points, we prove the existence and uniqueness of hyperbolic conformal metrics with prescribed singularities or degeneracies at the given points.

If $M$ is a closed Riemann surface with negative Euler characteristic $\chi(M)$, then it admits a compatible metric $g$ with Gauss curvature $K \equiv -1$. If $p \in M$, then we can ask for a compatible metric $\tilde{g}$ on $\bar{M} = M \setminus \{p\}$ with Gauss curvature $\tilde{K} \equiv -1$ and some prescribed singularity or degeneracy at $p$,

\begin{equation}
\frac{\tilde{g}}{g} = O(r^{2\alpha}) \quad \text{as} \quad r = r(x) = \text{dist}_g(x, p) \to 0.
\end{equation}

Such singularities arise, for example, from maps which are locally $z \to z^m (z \in \mathbb{C}, m \in \mathbb{Z}^+)$: pushing the standard metric forward gives a singularity corresponding to $\alpha = -(m-1)/m$ and pulling back the standard metric gives a degeneracy corresponding to $\alpha = m-1$. Thus we are particularly interested in (1) with $-1 < \alpha < \infty$.

More generally, we can consider a finite number of points $p_1, \ldots, p_n \in M$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and try to find a compatible metric $\tilde{g}$ on $\bar{M} = M \setminus \{p_1, \ldots, p_n\}$ with

\begin{equation}
\frac{\tilde{g}}{g} = O(r_i^{2\alpha_i}) \quad \text{as} \quad r_i = r_i(x) = \text{dist}_g(x, p_i) \to 0.
\end{equation}

Our main result is the following.

THEOREM. Let $(M, g)$ be a compact Riemann surface with Gauss curvature $K \equiv -1$ and $p_1, \ldots, p_n \in M$. Suppose the numbers $\alpha_1, \ldots, \alpha_n$ satisfy (i) $-1 < \alpha_i < \infty$, and (ii) $\chi(M) + \sum_i \alpha_i < 0$. Then $\bar{M} = M \setminus \{p_1, \ldots, p_n\}$ admits a unique metric $\tilde{g}$ which is pointwise conformal to $g$, has Gauss curvature $\tilde{K} \equiv -1$, and satisfies (2). Moreover, $\tilde{g}$ has total curvature

\begin{equation}
\int_M (-1) \, d\tilde{A} = 2\pi \left( \chi(M) + \sum_i \alpha_i \right).
\end{equation}

PROOF. We shall assume for notational convenience that $n = 1$ but all steps of the proof generalize immediately. We want to solve

\begin{equation}
\Delta u - e^{2u} = -1 \quad \text{on} \quad \bar{M} = M \setminus \{p\}
\end{equation}
with \( u \in C^\infty(M) \) of the form \( u = u_1 + v \) where \( u_1 \) is singular, behaving like \( \alpha \ln r \) as \( r \to 0 \), and \( v \) is bounded; in fact we will have \( v \in C^\beta(M) \) for some \( \beta > 0 \), indeed \( v \in C^2(M) \) if \( \alpha > 0 \). Then \( \hat{g} = e^{2u}g \) satisfies (1).

To define \( u_1 \), let \( \delta \) be so small that \( D = \{x \in M : r(x) < 2\delta \} \) is a disk and let \( G(x) \) be the Green’s function for \( D \) with singularity at \( p \), so \( G(x) = -(2\pi)^{-1} \ln(r(x)) + O(1) \) as \( r(x) \to 0 \). Choose \( u_1 \in C^\infty(M) \) so that \( u_1(x) = -2\pi \alpha G(x) \) for \( r(x) < \delta \). Notice that

\[
\int_M \Delta u_1 \, dA = -2\pi \alpha.
\]

We now must find a bounded solution \( v \) of

\[
\Delta v = H e^{2v} - \Delta u_1 - 1 \quad \text{on } M
\]

where \( H(x) = \exp[2u_1(x)] = O(r^{2\alpha}) \) as \( r \to 0 \). We can reduce this problem even further. Let \( u_0 \in C^\infty(M) \) satisfy

\[
\Delta u_0 = -1 - \Delta u_1 - 2\pi \alpha (r(M) + \alpha)
\]

(notice the right-hand side has integral zero). If we let \( w = v - u_0 \) and \( c = 2\pi (r(M) + \alpha) \), then we want to solve

\[
\Delta w = c + h e^{2w} \quad \text{on } M
\]

where \( h = H \exp[2u_0] > 0, h = O(r^{2\alpha}) \) as \( r \to 0 \), and \( c < 0 \) by (ii).

The form of this equation is familiar from [2], and the method of upper and lower solutions employed in [2] to solve it need only be modified slightly to allow the mild singularity of \( h \).

By (i) we find \( h \in L^p(M) \) for \( 1 \leq p < p_0 \) where \( p_0 = \infty \) (if \( \alpha > 0 \)) or \( p_0 = -1/\alpha \) (if \(-1 < \alpha < 0 \)). Thus we may define \( \bar{h} = (\int h \, dA) / (\int dA) \) and solve \( \Delta \varphi = h - \bar{h} \) with \( \varphi \in H^2_0(M) \) and so \( \varphi \in C^\beta(M) \) for some \( \beta > 0 \); in particular, \( \varphi \) is bounded. Choose \( a \) and \( b > 0 \) so that \( h > -c/a \) and \( \exp[2a \varphi + 2b] > a \). Then \( w_+ = a \varphi + b \) satisfies \( \Delta w_+ < c + h e^{2w_+} \). Now let \( k(x) = \max(1, h(x)) \) and \( \mu > 0 \) so that \( k = -c/\mu \). Then we can solve \( \Delta \psi = \mu k + c \) with \( \psi \in H^2_0(M) \) for all \( p \), and so \( \psi \in C^{1+\beta}(M) \) for all \( 0 < \beta < 1 \); in particular, \( \psi \) is bounded. Choose \( \lambda \) sufficiently large that \( \psi - \lambda \) satisfies \( w_- < w_+ \) and \( \Delta w_- > c + h e^{2w_-} \).

REMARK. If \( \alpha < 0 \) we could take \( w_+ = \) large positive constant whereas if \( \alpha > 0 \) then we could take \( w_- = \) large negative constant; however this fails if \( n > 1 \) and the \( \alpha_i \) having varying sign.

We cannot apply Lemma 9.3 of [2] directly since we will not have \( w_+ \in C^1(M) \) if \(-1 < \alpha \leq -1/2 \). However, \( w_+ \in C^1(M) \) so we can apply the monotone iteration scheme on a sequence of closed sets \( M_j \) with \( \hat{M} = \bigcup M_j \) (cf. the proof of Theorem 1 in [1]). We obtain a bounded solution \( w \) of (7). Since \( h \in C^\infty(M) \cap L^p(M) \) for \( 1 \leq p < p_0 \), elliptic regularity shows \( w \in C^\infty(M) \cap C^\beta(M) \) for some \( \beta > 0 \) (in fact \( w \in C^2(M) \) if \( \alpha > 0 \)). If we let \( S_\delta(p) = \{x \in M : r(x) = \delta \} \) and \( \nu \) denote the unit normal (towards \( p \)) then

\[
\int_M \Delta w \, dA = \lim_{\delta \to 0} \int_{S_\delta} \frac{\partial w}{\partial \nu} \, d\sigma = 0.
\]
Thus $u = u_1 + u_0 + w$ solves (4); moreover
\[
\int_M (-1) d\hat{A} = \int_M (-1)e^{2u} dA = \int_M (-1 - \Delta u) dA
\]
yields (3) by (5) and (8).

To verify uniqueness, suppose $\hat{g}_1$ and $\hat{g}_2$ are 2 such metrics. Since they both satisfy (1) this means that there are 2 solutions, $w_1$ and $w_2$, of (7). If $\alpha > 0$ then $w_1, w_2 \in C^2(M)$ so $w_1 = w_2$ by the maximum principle. Otherwise, let $w_0 = w_1 - w_2$ which satisfies $\Delta w_0 = h(e^{2u_1} - e^{2u_2})$ and $w_0 \in C^\infty(M) \cap C^2(M)$. Although $w_0$ achieves its maximum and minimum on $M$, it cannot achieve a positive maximum or negative minimum on $\hat{M}$ by the maximum principle. So suppose $w_0$ achieves a positive maximum at $p$: for some small neighborhood $U$ of $p$ we have $w_0(x) > 0$ for $x \in U$ and $\partial w_0/\partial \nu \leq 0$ on $\partial U$ where $\nu$ is the outward normal on $\partial U$. But, arguing as in (8), we find
\[
0 < \int_U \Delta w_0 dA = \int_{\partial U} \frac{\partial w_0}{\partial \nu} d\sigma \leq 0
\]
a contradiction. Similarly we find that we cannot have a negative minimum at $p$. Thus $w_1 = w_2$ and the metric $\hat{g}$ is unique.

**REMARK.** Note that the metric $\hat{g}$ is incomplete on $\hat{M}$ by (i). In fact, if $\alpha \leq -1$ then it can be shown (e.g. by an argument similar to the proof of Lemma B in [3]) that (6) admits no bounded solution. Hence condition (i) is necessary, i.e. $g$ admits no complete conformal metric $\hat{g}$ satisfying (1) and $\hat{K} \equiv -1$. On the other hand, by uniformization theory, it is known that $(\hat{M}, g)$ does admit a complete conformal metric $\hat{g}$ with $\hat{K} \equiv -1$. In fact, if we replace $g$ by a conformal metric which is Euclidean near $p$, i.e. $g = dx^2 + dy^2$ and $\hat{K} \equiv 0$ near $p$, then it is possible to solve $\Delta u - e^{2u} = K$ on $M$ with $u \geq -\ln r - \ln |\ln r| - C$ as $r = (x^2 + y^2)^{1/2} \to 0$ where $C$ is a constant. Thus (1) is replaced by $\hat{g}/g = O(r^{-2} |\ln r|^{-2})$ as $r \to 0$.

**ACKNOWLEDGMENTS.** The author wishes to thank K. Uhlenbeck and W. Goldman for bringing this problem to his attention, and UCSD where the work was conducted.

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