ABSTRACT. Let $Q \subseteq C$ be a bounded, simply connected domain, and let
$\{\Phi_n(w)\}_{n=0}^{\infty}$ be the Faber polynomials associated with $Q$. Given $f(z) = \sum_{k=0}^{\infty} c_k z^k$ analytic in $\Delta(0,1)$ we consider the function

$$F(w) = \sum_{k=0}^{\infty} c_k \Phi_k(w).$$

We show that with proper restrictions on $\partial Q$, the existence of an analytic continuation of $f$ across a subarc of $C(0,1)$ implies the existence of an analytic continuation of $F$ across a subarc of $\partial Q$. Some converse results are also established.

1. Introduction. Let $Q \subseteq C$ be a bounded, simply connected domain for which $C \setminus Q$ is connected. We use $g(\xi)$ to denote the unique function that is analytic and univalent on $\{|\xi| > 1\}$, maps $\{|\xi| > 1\}$ onto the exterior of $Q$, and has expansion

$$(1) \quad g(\xi) = b_{-1} \xi + b_0 + b_1/\xi + b_2/\xi^2 + \cdots \quad (b_{-1} > 0)$$

in a neighborhood of $\infty$. The Faber polynomials associated with $Q$ (or $g$) are the polynomials $\{\Phi_n(w)\}_{n=0}^{\infty}$ determined by the following generating function relationship $[P]$:

$$\frac{g'(\xi)}{g(\xi) - w} = \sum_{k=0}^{\infty} \Phi_k(w) \xi^{-k-1}.$$

It can be shown that with proper restrictions on $\partial D$ (see for example [S]) any function $F(w)$ analytic in $Q$ has a unique "Faber expansion"

$$(2) \quad \sum_{k=0}^{\infty} c_k \Phi_k(w)$$

that converges to $F(w)$ uniformly on compact subsets of $Q$. Conversely (again with appropriate restriction on $\partial Q$) any expression of the form (2) converges uniformly on compact subsets of $Q$ provided $\limsup_{k \to \infty} \sqrt[k]{|c_k|} \leq 1$.

In view of the last statement, we see that if

$$(3) \quad f(z) = \sum_{k=0}^{\infty} c_k z^k$$

is analytic on the unit disk $\Delta(0,1)$, then the function $F(w) = \sum_{k=0}^{\infty} c_k \Phi_k(w)$ might be analytic on $Q$. The operator $\mathcal{F}$ that takes a function (3) analytic on $\Delta(0,1)$ and maps it to the (formal) Faber series $\sum_{k=0}^{\infty} c_k \Phi_k(w)$ is called the Faber transform.

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In this paper we address some aspects of the following question: if \( f(z) = \sum_{k=0}^{\infty} c_k z^k \) is analytic on \( \Delta(0,1) \) and has a certain property \( P \), then to what extent is this property "inherited" by \( F(w) = \mathcal{F}(f(z))(w) = \sum_{k=0}^{\infty} c_k \Phi_k(w) \)? For example, it is clear that if \( f(z) \) is a polynomial, then so is \( \mathcal{F}(f(z))(w) \). Ellacott (and Gaier) [E] have shown that if \( f(z) \) is a rational function then the same is true of \( \mathcal{F}(f(z))(w) \) ... no restrictions on \( \partial \Omega \) are required. In this paper we consider the question of analytic continuation. We show that with proper restriction on \( \partial \Omega \) and/or on \( f(z) \), \( \mathcal{F}(f(z))(w) \) has analytic continuation properties similar to those of \( f(z) \).

In the remainder of the paper, \( g(\xi) \) will be as defined in (1). We will also assume \( b_{-1} = 1 \) ... this assumption simply results in a change of scale and has no affect on the results.

2. \( \partial \Omega \) analytic. As one would expect, the easiest case to consider is that in which \( \partial \Omega \) is analytic.

**THEOREM 1.** Let \( \partial \Omega \) be analytic and let \( J \subseteq \partial \Omega \) be a subarc of \( \partial \Omega \). For a given \( f(z) \) analytic on \( \Delta(0,1) \), the function \( F(w) = \mathcal{F}(f(z))(w) \) is analytic on \( \Omega \) and has an analytic continuation across \( J \) if and only if \( f(z) \) has an analytic continuation across \( g^{-1}(J) \).

**PROOF.** The fact that \( F(w) \) is analytic on \( \Omega \) is well known (see [S]). Since \( \partial \Omega \) is analytic, \( g(\xi) \) can be analytically and univalently continued to some domain \( \{ |\xi| > r_0 \} \) for some \( r_0 \in (0,1) \). Hence if \( f(z) = \sum_{k=0}^{\infty} c_k z^k \) on \( \Delta(0,1) \), then for \( r_0 < |\xi| < 1 \) we have

\[
F(g(\xi)) = \sum_{k=0}^{\infty} c_k \Phi_k(g(\xi))
\]

(4)

\[
= \sum_{k=0}^{\infty} c_k \left( \xi^k + k \sum_{l=1}^{\infty} b_{kl} \xi^{-l} \right)
\]

\[
= f(\xi) + \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} k c_k b_{kl} \xi^{-l}
\]

where the coefficients \( \{ b_{kl} \}_{k,l=1}^{\infty} \) are the Grunsky coefficients associated with \( g \) (see [P]). Since \( g(\xi) \) is analytic and univalent in \( \{ |\xi| > r_0 \} \), it follows that for any \( r \in (r_0,1) \), we have \( |b_{kl}| = o(r^{k+l}) \) as \( k + l \to \infty \). Thus the last sum in (4) defines a function analytic in \( \{ |\xi| > r_0 \} \). Hence, if \( f(\xi) \) can be continued analytically across \( g^{-1}(J) \subseteq \{ |\xi| = 1 \} \), then the same is true of \( F(g(\xi)) \). Let \( h(\xi) \) denote such an analytic continuation. Then for some open neighborhood \( N(g^{-1}(J)) \subseteq \{ |\xi| > r_0 \} \) of \( g^{-1}(J) \) we have \( h(\xi) \) analytic on \( \{ r_0 < |\xi| < 1 \} \cup N(g^{-1}(J)) \) and \( h(\xi) \equiv F(g(\xi)) \) on \( \{ r_0 < |\xi| < 1 \} \). Define

\[
\tilde{F}(w) = \begin{cases} 
F(w) & (w \in \Omega), \\
h(g^{-1}(w)) & (w \in g(\{ r_0 < |\xi| < 1 \} \cap N(g^{-1}(J)))).
\end{cases}
\]

Then \( \tilde{F}(w) \) is an analytic continuation of \( F \) across \( J \).

Conversely, if \( F(w) \) has an analytic continuation across \( J \), then \( F(g(\xi)) \), defined for \( r_0 < |\xi| < 1 \), has an analytic continuation across \( g^{-1}(J) \). It then follows from (4) that \( f(z) \) has an analytic continuation across \( g^{-1}(J) \). \( \Box \)
In view of Theorem 1, we see that results concerning analytic continuations of functions analytic on $\Delta(0,1)$ have counterparts for functions defined by Faber expansions. For example, we have a “Faber-Hadamard” gap theorem.

**Corollary.** Let $\partial \Omega$ be analytic and let

$$F(w) = \sum_{k=1}^{\infty} c_{n_k} \Phi_{n_k}(w)$$

be analytic in $\Omega$, but not on any neighborhood of $\overline{\Omega}$. Suppose there is a $\lambda > 1$ such that the integer sequence $\{n_k\}_{k=1}^{\infty}$ satisfies $n_{k+1}/n_k \geq \lambda$ ($k \geq k_0$). Then $\partial \Omega$ is the natural boundary of $F$.

### 3. Nonanalytic $\partial \Omega$.

If $\partial \Omega$ is not analytic, then there is no guarantee that $F(w) = \mathcal{F}(f(z))(w)$ defines a function analytic on $\Omega$. Something can be said, however, in the case where $\partial \Omega$ is a curve of bounded rotation and $f(z) \in A(\Delta(0,1)) = \{f(z) : f$ analytic on $\Delta(0,1)$, continuous on $\Delta(0,1)\}$. Even if $\partial D$ is of bounded rotation and $f \in A(\Delta(0,1))$ with

$$f(z) = \sum_{k=0}^{\infty} c_k z^k,$$

it is possible that $\sum_{k=0}^{\infty} c_k \Phi_k(w)$ does not define a function in $A(\overline{\Omega})$. However if, in this case, we define

$$F(w) = \mathcal{F}(f(z))(w) \equiv \frac{1}{2\pi i} \int_{\partial \Omega} \frac{f(g^{-1}(\xi))}{\xi - w} d\xi \quad (w \in \Omega),$$

then $F \in A(\overline{\Omega})$ and has Faber coefficients $\{c_k\}_{k=0}^{\infty}$ agreeing with the Maclaurin coefficients of $f$ (see [G]). In this case we use the integral expression

$$c_k = \frac{1}{2\pi i} \int_{|z|=1} \frac{F(g(z))}{z^{n+1}} dz$$

for the coefficients. We remark that it is easily checked that (5) agrees with the “coefficient transplant” description of $\mathcal{F}$ in the case when $\partial \Omega$ is analytic.

Thus if $\partial \Omega$ is of bounded rotation, then (5) defines a linear mapping

$$\mathcal{F} : A(\Delta(0,1)) \to A(\overline{\Omega}).$$

In fact, $\mathcal{F}$ is a continuous operator with respect to the supremum norms on $A(\Delta(0,1))$ and $A(\overline{\Omega})$, with $\| \mathcal{F} \| \leq (1 + 2V/\pi)$, where $V$ is the total rotation of $\partial \Omega$. (See [G].)

**Theorem 2.** Let $\partial \Omega$ be of bounded rotation and let $f \in A(\Delta(0,1))$. If $f(z)$ has an analytic continuation across the arc $I \subseteq C(0,1)$, then $F(w) = \mathcal{F}(f(z))(w)$ has an analytic continuation across $g(I)$.

**Proof.** We first assume $I$ is a closed subarc of the unit circle, i.e. $I = \{e^{i\theta} : \alpha \leq \theta \leq \beta\}$. Let $h(z)$ be an analytic continuation of $f$ to $\Delta(0,1) \cup N(I)$ where $N(I)$ is some open neighborhood of $I$. Since $I$ is closed, we may assume $h$ is defined and continuous on $\Delta(0,1) \cup N(I)$ and that $C \setminus \Delta(0,1) \cup N(I)$ is connected. By Mergelyan’s Theorem [R] there is a sequence $\{P_n(z)\}_{n=1}^{\infty}$ of polynomials such that
$P_n(z) \to h(z)$ uniformly on $\overline{\Delta(0,1) \cup N(I)}$. Now consider the sequence of polynomials

$$Q_n(w) = \mathcal{F}(P_n(z))(w) \quad (n = 1, 2, 3, \ldots).$$

For $w \in \overline{\Omega}$ we have

$$|Q_n(w) - F(w)| = \left| \mathcal{F}(P_n(z))(w) - \mathcal{F}(f(z))(w) \right| \\
\leq \|\mathcal{F}\| \left\{ \sup_{|z| \leq 1} |P_n(z) - f(z)| \right\} \\
= \|\mathcal{F}\| \left\{ \sup_{|z| \leq 1} |P_n(z) - h(z)| \right\} \\
\to 0 \quad \text{as } n \to \infty.$$

(6)

As equation (6) shows, $\{Q_n(w)\}_{n=1}^{\infty}$ converges uniformly to $F$ on $\overline{\Omega}$. We claim that $\{Q_n(w)\}_{n=1}^{\infty}$ also converges uniformly on $g(N(I) \cap \{|\xi| \geq 1\})$.

Given $w \in g(N(I) \cap \{|z| > 1\})$, find $\xi \in \overline{N(I)} \cap \{|z| > 1\}$ with $w = g(\xi)$. Writing

$$P_n(\xi) = \sum_{k=0}^{m_n} c_k^{(n)} \xi^k,$$

we have

$$Q_n(w) = \sum_{k=0}^{m_n} c_k^{(n)} \Phi_k(w) = \sum_{k=0}^{m_n} c_k^{(n)} \Phi_k(g(\xi)) \\
= \sum_{k=0}^{m_n} c_k^{(n)} \xi^n + \sum_{k=0}^{m_n} \left( k c_k^{(n)} \sum_{l=1}^{\infty} b_{kl} \xi^{-l} \right) \\
= P_n(\xi) + \sum_{l=1}^{\infty} \left( \sum_{k=0}^{m_n} k c_k^{(n)} b_{kl} \right) \xi^{-l}.$$

Now

$$R_n(\xi) = Q_n(g(\xi)) - P_n(\xi) = \sum_{l=1}^{\infty} \left( \sum_{k=0}^{m_n} k c_k^{(n)} b_{kl} \right) \xi^{-l}$$

is analytic in $|\xi| > 1$ and vanishes at $\infty$. Furthermore, for a given real $\theta$,

$$\lim_{\xi \to e^{i\theta}} R_n(\xi) = Q_n(g(e^{i\theta})) - P_n(e^{i\theta}).$$

Since $\{P_n\}$ is uniformly Cauchy on $\overline{\Delta(0,1)}$ and $\{Q_n(w)\}$ is uniformly Cauchy on $\overline{\Omega}$, it follows from the Maximum Modulus Theorem that $\{R_n(\xi)\}$ is uniformly Cauchy on $\{|\xi| \geq 1\}$. Now for $w = g(\xi) \in g(N(I) \cap \{|z| > 1\})$, we have

$$|Q_n(w) - Q_m(w)| = |Q_n(g(\xi)) - Q_m(g(\xi))| \\
\leq |P_n(\xi) - P_m(\xi)| + |R_n(\xi) - R_m(\xi)|.$$

Since $\{P_n(\xi)\}$ is uniformly Cauchy on $\overline{\Delta(0,1) \cup N(I)}$ and $\{R_n(\xi)\}$ is uniformly Cauchy on $\{|\xi| \geq 1\}$, it follows that $\{Q_n(w)\}$ is uniformly Cauchy in $g(N(I) \cap \{|\xi| \geq 1\})$. Combining with (6) we see that $\{Q_n(w)\}$ is uniformly Cauchy on $\overline{\Omega} \cup g(N(I) \cap \{|\xi| \geq 1\})$... the continuity of $g$ on $\{|\xi| \geq 1\}$ and the analyticity...
of \( g \) on \( \{|\xi| > 1\} \) imply this last set is a neighborhood of the (closed) arc \( g(I) \).

Letting
\[
H(w) = \lim_{n \to \infty} Q_n(w)
\]
we have a function continuous on \( \overline{\Omega} \cup g(N(I) \cap \{|\xi| \geq 1\}) \) and analytic on the interior of this set. Since this interior contains \( g(I) \) and since \( H|_{\overline{\Omega}} = F \), \( H \) is the desired analytic continuation.

Suppose now the subarc \( I \) of the unit circle is not closed. Again let \( N(I) \) be an open set containing \( I \) with \( C \setminus (\Delta(0,1) \cup N(I)) \) connected and suppose \( h(z) \) is an analytic continuation of \( f \) to \( \Delta(0,1) \cup N(I) \). We write \( I = \bigcup_{n=1}^{\infty} I_n \) where \( \{I_n\}_{n=1}^{\infty} \) is an increasing sequence of closed subarcs of \( I \). We can also find an increasing sequence \( \{N(I_n)\}_{n=1}^{\infty} \) of open sets satisfying \( I_n \subset N(I_n) \subset N(I_{n+1}) \) (\( n = 1, 2, \ldots \)), with each \( C \setminus (\Delta(0,1) \cup N(I)) \) connected and \( \bigcup_{n=1}^{\infty} N(I_n) = N(I) \). Then \( h_n(z) = h|_{\Delta(0,1) \cup N(I_n)} \) is an analytic continuation of \( f \) across \( I_n \), and has a continuous extension to \( \Delta(0,1) \cup N(I_n) \). As shown in the first part of the proof, we can find, for each \( n \), a function \( H_n(w) \) continuous on \( \overline{\Omega} \cup g(N(I_n) \cap \{|\xi| \geq 1\}) \), analytic on the interior of this set and with \( H_n|_{\overline{\Omega}} = F \). Since

\[
\Omega \cup g(N(I) \cap \{|\xi| \geq 1\}) = \bigcup_{n=1}^{\infty} [\Omega \cup g(N(I_n) \cap \{|\xi| \geq 1\})]
\]
as an increasing union, we may define \( H \) on \( \Omega \cup \{g(N(I) \cap \{|\xi| \geq 1\}) \} \) by
\[
H(w) = \begin{cases} 
F(w) & (w \in \Omega), \\
H_n(w) & (w \in g(N(I_n) \cap \{|\xi| \geq 1\})).
\end{cases}
\]
It is clear that \( H \) is well defined and analytic on \( \Omega \cup \{g(N(I) \cap \{|\xi| \geq 1\}) \} \) and hence is the desired analytic continuation. \( \square \)

If \( \mathcal{F}^{-1}: A(\overline{\Omega}) \to A(\Delta(0,1)) \) is defined and continuous, then the converse of Theorem 2 holds. Unfortunately, the existence of \( \mathcal{F}^{-1} \) is not automatic, even if \( \partial \Omega \) is of bounded rotation. If \( \partial \Omega \) is of bounded rotation, then \( \mathcal{F}: A(\Delta(0,1)) \to A(\overline{\Omega}) \) is a one-to-one mapping, but may not be onto. In fact, given \( F \in A(\overline{\Omega}) \) (\( \partial \Omega \) of bounded rotation), one can assert that \( F = \mathcal{F}(f) \) for some \( f \in A(\Delta(0,1)) \) if and only if \( h = F \circ g \) and its conjugate, \( \tilde{h} \), are both continuous on \( C(0,1) \) (see [G, p. 53]). In this case \( F = \mathcal{F}(f) \) where
\[
f(z) = \frac{1}{2\pi i} \int_{|\xi|=1} \frac{F(g(\xi))}{\xi - z} d\xi.
\]
However, if \( \mathcal{F}^{-1}: A(\overline{\Omega}) \to A(\Delta(0,1)) \) is defined (i.e. \( \mathcal{F}: A(\Delta(0,1)) \to A(\overline{\Omega}) \) is onto), then \( \mathcal{F}^{-1} \) is continuous by the Open Mapping Theorem [R].

**Theorem 3.** Let \( \partial \Omega \) be of bounded rotation and let \( \mathcal{F}: A(\Delta(0,1)) \to A(\overline{\Omega}) \) be onto. Let \( F(w) \in A(\overline{\Omega}) \) and \( J \) be a subarc of \( \partial \Omega \). Then \( F(w) \) has an analytic continuation across \( J \) if and only if \( f(z) = \mathcal{F}^{-1}(F(w))(z) \) has an analytic continuation across \( g^{-1}(J) \).

**Proof.** The sufficiency was established in Theorem 2. The proof of the necessity is very similar so we may be somewhat brief. We consider the case in which \( J \) is a "closed subarc" of \( \partial \Omega \) ... that is \( J = g(\{e^{i\theta} : \alpha \leq \theta \leq \beta\}) \) for some real \( \alpha \) and \( \beta \).
The cases of "open" or "half-open" $J$ are then taken care of as in the last part of
the proof of Theorem 2.

Let $N(J)$ be an open set containing $J$ such that $C \setminus \Omega \cup N(J)$ is connected,
$F(w)$ has an analytic continuation, $H(w)$, to $\Omega \cup N(J)$ and $H$ has a continuous
extension to $\Omega \cup N(J)$.

By Mergelyan’s Theorem, there is a sequence \{\(Q_n(w)\)\} of polynomials with
$Q_n(w) \to H(w)$ uniformly on $\Omega \cup N(J)$. Now each $P_n(z) = \mathcal{F}^{-1}(Q_n(w))(z)$
is also a polynomial, and the continuity of $\mathcal{F}^{-1}$ implies that $P_n(z) \to f(z) =
\mathcal{F}^{-1}(F(w))(z)$ uniformly on $\Delta(0,1)$. We claim that the sequence \{\(P_n(z)\)\} is also
uniformly convergent on $g^{-1}(\overline{N(J)} \setminus (C \setminus \Omega))$. To see this write

$$Q_n(w) = \sum_{k=0}^{m_n} c_k^{(n)} \Phi_k(w)$$

as a Faber expansion. Then

$$P_n(z) = \sum_{k=0}^{m_n} c_k^{(n)} z^k.$$

Let $z \in \{|\xi| > 1\}$, $|z|$ large. We can find $w \in C \setminus \Omega$ ($|w|$ large) with $g^{-1}(z) = w$. Then

$$P_n(z) = P_n(g^{-1}(w)) = \sum_{k=0}^{m_n} c_k^{(n)}(g^{-1}(w))^k$$

$$= \sum_{k=0}^{m_n} c_k^{(n)} \left[ \Phi_k(w) + \sum_{l=1}^{\infty} d_{kl} w^{-l} \right]$$

$$= Q_n(w) + \sum_{l=1}^{\infty} \left( \sum_{k=0}^{m_n} c_k^{(n)} d_{kl} \right) w^{-l}$$

for some choice of coefficients $d_{kl}$ (see [G]). We note that

$$R_n(z) = P_n(z) - Q_n(g(z)) = \sum_{l=1}^{\infty} \left( \sum_{k=0}^{m_n} c_k^{(n)} d_{kl} \right) (g(z))^{-l}$$

(as above, defined for $|z|$ large) can be continued analytically to $\{|z| > 1\}$, and
continuously to $\{|z| \geq 1\}$. Furthermore, $R_n(z)$ vanishes at $\infty$. Since $P_n(z)$ and
$Q_n(g(z))$ are both uniformly Cauchy on $C(0,1)$ it follows from the Maximum Mod-
ulus Theorem that $\{R_n(z)\}$ is uniformly Cauchy on $\{|z| \geq 1\}$. From this point we
may proceed as in the proof of Theorem 2 and assert that

$$h(z) = \lim_{n \to \infty} P_n(z) = \begin{cases} f(z) & (z \in \Delta(0,1)), \\ \lim_{n \to \infty} (R_n(z) + Q_n(g(z))) & (z \in g^{-1}(N(J) \setminus (C \setminus \Omega))) \end{cases}$$

gives an analytic continuation of $f$ across $g^{-1}(J)$. \(\square\)

We conclude by noting that Theorem 3 allows us to state a "Faber-Hadamard"
gap theorem for $A(\overline{\Omega})$.

**COROLLARY.** Suppose $\partial \Omega$ is of bounded rotation and $\mathcal{F} : A(\overline{\Delta(0,1)}) \to A(\overline{\Omega})$
is onto. Let $F(w) \in A(\overline{\Omega})$ have Faber series

$$F(w) \sim \sum_{k=0}^{\infty} c_{nk} \Phi_{nk}(w).$$
If $\limsup_{k \to \infty} |c_{n_k}|^{1/n_k} = 1$ and there is a number $\lambda > 1$ with $n_{k+1}/n_k > \lambda$ ($k > k_0$), then $\partial \Omega$ is the natural boundary for $F$.

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