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POLYNOMIALLY MOVING ERGODIC AVERAGES

MARK SCHWARTZ

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ABSTRACT. Given an increasing sequence of positive integers $\{m_n\}$, a nondecreasing sequence of positive integers $\{b_n\}$, and a measurable, measure-preserving ergodic transformation τ on a probability space $(\Omega, \mathcal{F}, \mu)$, the a.s. convergence of the moving averages $T_n(f) = b_n^{-1} \sum_{k=m_n+1}^{m_n+b_n} f(\tau^k)$ is considered, for $f \in L_p(\Omega)$. A counterexample is constructed in the case of polynomial-like $\{m_n\}$.

In the paper of del Junco and Rosenblatt [2], it is shown that if $\{b_n\}$ is a nondecreasing sequence of positive integers satisfying $b_n/n \to 0$, and τ is an invertible measure-preserving ergodic transformation of a probability space $(\Omega, \mathcal{F}, \mu)$ then there exists $f \in L_p(\Omega)$, $1 \leq p < \infty$, such that the averages $T_n(f) = b_n^{-1} \sum_{k=n+1}^{n+b_n} f(\tau^k \omega)$ do not converge a.s. In fact, it is shown that there exists a dense G_δ subset $\mathcal{R}$ such that if $A \in \mathcal{R}$ then

$$\limsup_{n\to\infty} b_n^{-1} \sum_{k=n+1}^{n+b_n} I_A(\tau^k \omega) = 1 \quad \text{a.s.}$$

and

$$\liminf_{n\to\infty} b_n^{-1} \sum_{k=n+1}^{n+b_n} I_A(\tau^k \omega) = 0 \quad \text{a.s.}$$

This is an improvement of the results in [1] and [3].

Of course, for any $f \in L_p$, $1 \leq p < \infty$, the sequence $\{T_n(f)\}$ converges in L_p to $E(f) = \int_\Omega f \, d\mu$. Thus, there exists a subsequence $\{T_{k_n}(f)\}$ which converges a.s. to $E(f)$. Rosenblatt [4] poses the following question: does there exist a subsequence $\{k_n\}$ such that $T_{k_n}(f)\}$ converges a.s. to $E(f)$ for all $f \in L_1$? Naturally, the belief is that this is false, but apparently, it has never been proved.

In this paper, an extension of the construction in [3] is used to provide a counterexample for polynomial-like sequences $\{m_n\}$. Initially, it was believed that a generalization of this type, using Rohlin's lemma, could give counterexamples for any sequence $\{m_n\}$. However, it appears that a new approach is needed. We now state and prove the theorem for polynomial-like sequences.

Let $\{m_n\}$ be an increasing sequence of positive integers satisfying

$$\lim_{n\to\infty} m_{n-1}/m_n = 1.$$

Assume $\{b_n\}$ is a nondecreasing sequence of positive integers such that $\lim_{n\to\infty} b_n = \infty$ and $\lim_{n\to\infty} b_n/m_n = 0$.

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THEOREM. *Suppose $\{m_n\}$ and $\{b_n\}$ are as above and $\varepsilon > 0$. Then there exists a set $A \in \mathscr{F}$, $\mu(A) \le \varepsilon$, such that the sequence of averages $b_n^{-1}\sum_{k=m_n+1}^{m_n+b_n} I_A(\tau^k\omega)$ does not converge a.s.*

PROOF. For $f \in L_p$, $1 \le p < \infty$, denote $T_n(f) = b_n^{-1}\sum_{k=m_n+1}^{m_n+b_n} f(\tau^k\omega)$. For each $n \ge 2$, let $d_n = m_n - m_{n-1}$. With no loss of generality, we can assume $\{d_n\}$ is nondecreasing. For otherwise, we define a subsequence $\{m_{k_n}\}$ as follows: set $k_1 = 1$, $k_2 = 2$; for $i > 2$, define $k_i = \min\{j > k_{i-1} : m_j \ge m_{k_{i-1}} + d'_{i-1}\}$, where $d'_{i-1} = m_{k_{i-1}} - m_{k_{i-2}}$. Obviously, $\{d'_n\}$ is nondecreasing, and it can be shown that $\lim_{n\to\infty} m_{k_{n-1}}/m_{k_n} = 1$ from the corresponding property of $\{m_n\}$. For simplicity, the original notation is retained, even if we have passed to subsequences. Specifically, we write $\{m_n\}$, $\{b_n\}$, and $\{d_n\}$ for $\{m_{k_n}\}$, $\{b_{k_n}\}$, and $\{d'_n\}$, respectively.

By hypothesis, $\lim_{n\to\infty} d_n/m_n = 0$. For $p = 1, 2, \ldots$, let $n_p > n_{p-1}$ (taking $n_0 = 0$) be chosen such that

$$\frac{b_{n_p} + d_{n_p}}{m_{n_p} + b_{n_p}} < \varepsilon_p = \frac{\varepsilon}{2^p}$$

and

$$\frac{m_p}{m_{n_p}} < \frac{1}{p}.$$

By Rohlin's lemma, there exists $E_p \in \mathscr{F}$ such that $\{\tau^k E_p : 1 \le k \le m_{n_p} + b_{n_p}\}$ are disjoint and $\mu\bigcup_{k=1}^{m_{n_p}+b_{n_p}} \tau^k E_p \ge 1 - \varepsilon_p$. Define $A_p = \bigcup_{k=m_{n_p-1}+1}^{m_{n_p}+b_{n_p}} \tau^k E_p$; we have $\mu A_p \le (d_{n_p}+b_{n_p})/(m_{n_p}+b_{n_p}) < \varepsilon_p$. For $p \le j \le n_p - 1$, let $I_{p,j} = \{i : m_{n_p-1} - m_j \le i \le m_{n_p} + b_{n_p} - m_j - b_j\}$. Let $I_p = \bigcup_{j=p}^{n_p-1} I_{p,j}$ and define $D_p = \bigcup_{k\in I_p} \tau^k E_p$. Note, for $p \le j \le n_p - 1$, we have $m_{n_p} + b_{n_p} - m_{j+1} - b_{j+1} \ge m_{n_p-1} - m_j$, since $\{d_n\}$ is nondecreasing. So $I_{p,j+1}$ and $I_{p,j}$ overlap for $p \le j \le n_p - 1$. We compute

$$|I_p| = |[m_{n_p-1} - m_{n_p-1}, m_{n_p} + b_{n_p} - m_p - b_p]| \ge m_{n_p} + b_{n_p} - m_p - b_p.$$

Then,

$$\begin{aligned}\mu D_p &\ge |I_p|(1-\varepsilon_p)/(m_{n_p} + b_{n_p}) \\ &\ge (m_{n_p} + b_{n_p} - m_p - b_p)(1-\varepsilon_p)/(m_{n_p} + b_{n_p}).\end{aligned}$$

Consequently, $\lim_{p\to\infty} \mu D_p = 1$. Suppose $\omega \in D_p$. Then there exists $i \in I_p$ such that $\omega \in \tau^i E_p$. Further, there exists j, $p \le j \le n_p - 1$, for which $i \in I_{p,j}$. Then, for $m_j + 1 \le k \le m_j + b_j$, we have $\tau^k\omega \in A_p$. Thus, for each $\omega \in D_p$, there exists $j \ge p$ such that $b_j^{-1}\sum_{k=m_j+1}^{m_j+b_j} I_{A_p}(\tau^k\omega) = 1$. Let $A = \bigcup_{p=1}^{\infty} A_p$, and $D = \bigcup_{p=1}^{\infty}\bigcup_{j=p}^{\infty} D_j$. We have $\mu A \le \sum_{p=1}^{\infty} \varepsilon/2^p = \varepsilon$, and $\mu D = 1$.

For each $\omega \in D$,

$$\limsup_{n\to\infty} b_n^{-1} \sum_{k=m_n+1}^{m_n+b_n} I_A(\tau^k\omega) = 1.$$

However, if $T_n(I_A) \to g$ a.s. then $E(T_n(I_A)) \to E(g)$. But, $E(T_n(I_A)) = \mu A \le \varepsilon$. Thus, $\{T_n(I_A)\}$ does not converge a.s.

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DEPARTMENT OF MATHEMATICAL SCIENCES, OHIO WESLEYAN UNIVERSITY, DELAWARE, OHIO 43015