THE CARDINALITY OF REDUCED POWER SET ALGEBRAS

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ABSTRACT. We prove a general result on the cardinality of reduced powers of structures via filters that has several consequences including the following: if $I$ is a uniform, countably complete ideal on the real line $\mathbb{R}$ and $\mathcal{B}$ is the Boolean algebra of subsets of $\mathbb{R}$ modulo $I$, then $|\mathcal{B}| > 2^{\aleph_0}$ and if $2^{\nu} \leq 2^{\aleph_0}$ for all $\nu < 2^{\aleph_0}$ then $|\mathcal{B}| = 2^{2^{\aleph_0}}$. This strengthens some results of Kunen and Pele [7] and Prikry [8] obtained by Boolean ultrapower techniques. Our arguments are all combinatorial and some applications are included.

1. Introduction. Our notation and terminology are fairly standard. If $A$ and $B$ are sets then $A B$ denotes the set of all functions mapping $A$ into $B$ and $\mathcal{P}(A)$ denotes the power set of $A$. Throughout, $\kappa$ and $\lambda$ will denote infinite cardinals. An ideal $I$ on $\kappa$ is a collection of subsets of $\kappa$ that is closed under subset formation and finite unions. All the ideals we will consider are assumed to be proper ($I \neq \mathcal{P}(\kappa)$) but not necessarily $\kappa$-complete (i.e. closed under unions of size less than $\kappa$) or even uniform (i.e. containing all subsets of $\kappa$ of size less than $\kappa$). If $I$ is an ideal on $\kappa$ then $I^*$ denotes the filter on $\kappa$ that is dual to $I$; that is, $X \in I^*$ iff $\kappa - X \in I$, and $I^+$ denotes $\mathcal{P}(\kappa) - I$.

If $I$ is an ideal on $\kappa$ and $B$ is an arbitrary set then the reduced power of $B$ (via $I^*$), denoted $\prod_{\kappa} B/I^*$, is the set of all equivalence classes of functions in $B^\kappa$ where we identify two such functions $f$ and $g$ if $\{\alpha < \kappa : f(\alpha) \neq g(\alpha)\} \in I$. The Boolean algebra $\mathcal{P}(\kappa)/I$ is the set of all equivalence classes of subsets of $\kappa$ with the induced ordering from $\subseteq$, where we identify two such subsets $X$ and $Y$ if their symmetric difference $X \triangle Y$ is in $I$.

Since we will be comparing $\mathcal{P}(\kappa)/I$ and $\prod_{\kappa} \kappa/I^*$ it is worth pointing out the following (see [3 or 1, p. 338]). The function which associates to each $X \subseteq \kappa$ its characteristic function induces a natural bijection between $\mathcal{P}(\kappa)/I$ and $\prod_{\kappa} 2/I^*$ where, of course, $2 = \{0,1\}$. Hence $|\mathcal{P}(\kappa)/I| = |\prod_{\kappa} 2/I^*| \leq |\prod_{\kappa} \kappa/I^*|$. In general, the inequality cannot be replaced by equality since trivially $|\prod_{\kappa} \kappa/I^*| \geq \kappa$ (consider the constant functions from $\kappa$ to $\kappa$) but $|\mathcal{P}(\kappa)/I| = 2$ if $I$ is a prime (i.e. maximal) ideal on $\kappa$.

2. The main results. At the heart of our first theorem is the following trivial observation: there is a natural bijection between $\kappa$-sequences of subsets of $\lambda$ and $\lambda$-sequences of subsets of $\kappa$. This correspondence is given by $(X_\alpha : \alpha < \kappa) \leftrightarrow (Y_\beta : \beta < \lambda)$ where $\alpha \in Y_\beta$ iff $\beta \in X_\alpha$. The proof of the following result consists of simply verifying that if $I$ is a $\lambda^+$-complete ideal on $\kappa$ then this correspondence induces a (well-defined) bijection between $\prod_{\kappa} \mathcal{P}(\lambda)/I^*$ and $^\lambda(\mathcal{P}(\kappa)/I)$.

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THEOREM 2.1. If $I$ is a $\lambda^+$-complete ideal on $\kappa$ then
$$\prod_{\kappa}^{\lambda^2/I^*} = \left( \prod_{\kappa}^{\lambda^2} \right)^{\lambda}.$$ 

PROOF. Define $\phi: \prod_{\kappa}^{\lambda^2/I^*} \to \lambda\left( \prod_{\kappa}^{\lambda^2/I^*} \right)$ as follows: if $[f] \in \prod_{\kappa}^{\lambda^2/I^*}$, then $f: \kappa \to \lambda^2$ and we set
$$\phi([f]) = \langle [f_\beta]: \beta < \lambda \rangle$$
where $f_\beta: \kappa \to 2$ is given by $f_\beta(\alpha) = f(\alpha)(\beta)$. Note that $[f_\beta] \in \prod_{\kappa}^{2/I^*}$.

To complete the proof of the theorem it suffices to show that $\phi$ is a well-defined bijection.

CLAIM 1. $\phi$ is well defined.

PROOF. Suppose that $f, g: \kappa \to \lambda^2$ and $[f] = [g]$. Let $X = \{ \alpha < \kappa: f(\alpha) = g(\alpha) \}$. Then $X \subseteq I^*$. Fix $\beta < \lambda$ and let $Y_\beta = \{ \alpha < \kappa: f_\beta(\alpha) = g_\beta(\alpha) \}$. We claim that $X \subseteq Y_\beta$. That is, if $\alpha \in X$ then $f(\alpha) = g(\alpha)$ so $f(\alpha)(\beta) = g(\alpha)(\beta)$. But then $f_\beta(\alpha) = g_\beta(\alpha)$ so $\alpha \in Y_\beta$ as desired. But now since $X \subseteq I^*$ and $X \subseteq Y_\beta$, we have $Y_\beta \subseteq I^*$ and so $[f_\beta] = [g_\beta]$. Since $\beta$ was arbitrary this shows that $\langle [f_\beta]: \beta < \lambda \rangle = \langle [g_\beta]: \beta < \lambda \rangle$. Thus $\phi([f]) = \phi([g])$ and so $\phi$ is well defined as claimed.

CLAIM 2. $\phi$ is one-to-one.

PROOF. Suppose that $f, g: \kappa \to \lambda^2$ and assume that $\phi([f]) = \phi([g])$. Then for every $\beta < \lambda$ we have $[f_\beta] = [g_\beta]$ and so $Y_\beta \subseteq I^*$ where $Y_\beta = \{ \alpha < \kappa: f_\beta(\alpha) = g_\beta(\alpha) \}$. Let $Y = \bigcap_{\beta < \lambda} Y_\beta$. Since $I$ is $\lambda^+$-complete we have $Y \subseteq I^*$. Now let $X = \{ \alpha < \kappa: f(\alpha) = g(\alpha) \}$. We claim that $X \subseteq Y$. That is, if $\alpha \in X$ then for every $\beta < \lambda$ we have $\alpha \in Y_\beta$, and so $f_\beta(\alpha) = g_\beta(\alpha)$. But then for every $\beta < \lambda$ we have $f(\alpha)(\beta) = g(\alpha)(\beta)$ and so the functions $f(\alpha)$ and $g(\alpha)$ are equal. Hence $\alpha \in X$. Now, since $Y \subseteq I^*$ and $Y \subseteq X$ we have $X \subseteq I^*$ and so $[f] = [g]$ as desired.

CLAIM 3. $\phi$ is onto.

PROOF. Suppose that $\beta < \lambda$. Define $f: \kappa \to \lambda^2$ by $f(\alpha)(\beta) = f_\beta(\alpha)$. Then $[f] \in \prod_{\kappa}^{\lambda^2/I^*}$ and clearly $\phi([f]) = \langle [f_\beta]: \beta < \lambda \rangle$. □

Specializing Theorem 2.1 to the case where $\kappa = 2^\lambda$ and making use of some known results yields the following.

THEOREM 2.2. Suppose $I$ is a $\lambda^+$-complete ideal on $\kappa = 2^\lambda$. Then

(i) $|\mathcal{P}(\kappa)/I|^\lambda = |\prod_{\kappa}^{\kappa/I^*}| \geq \kappa$, and

(ii) if $I$ is also uniform (i.e. $[\kappa]^{<\kappa} \subseteq I$), then $|\mathcal{P}(\kappa)/I| > \kappa$.

PROOF. Part (i) follows immediately from Theorem 2.1 and our comments regarding $\mathcal{P}(\kappa)/I$ and $\prod_{\kappa}^{2/I^*}$. For part (ii) we need the following well-known result (see [3, p. 204]): for every infinite cardinal $\gamma$ there exists a family $\mathcal{F} \subseteq \gamma^\gamma$ with $|\mathcal{F}| = \gamma^+$ and $|\{ \alpha < \gamma: f(\alpha) = g(\alpha) \}| < \gamma$ whenever $f, g \in \mathcal{F}$ and $f \neq g$.

An immediate consequence of this is the fact that if $I$ is a uniform ideal on $\kappa$ then $|\prod_{\kappa}^{\kappa/I^*}| \geq \kappa^+$. To complete the proof of (ii), let $\mu = |\mathcal{P}(\kappa)/I|$ and suppose for contradiction that $\mu \leq \kappa$. Then
$$\kappa^+ \leq \left| \prod_{\kappa}^{\kappa/I^*} \right| = |\mathcal{P}(\kappa)/I|^\lambda = \mu^\lambda \leq \kappa^\lambda = (2^\lambda)^\lambda = 2^\lambda = \kappa.$$ □

Finally, if we specialize still further and again use some known results we obtain the following.
THEOREM 2.3. Suppose \( I \) is an \( \omega_1 \)-complete ideal on the real line \( \mathbb{R} \). Then

(i) \( |\mathcal{P}(\mathbb{R})/I| = \prod_{\alpha \in \mathcal{I}} |\mathcal{P}(\mathcal{B})/I^*|^{\omega} \),

(ii) if \( I \) is uniform then \( |\mathcal{P}(\mathcal{B})/I| > 2^{\aleph_0} \), and

(iii) if \( I \) is uniform and \( 2^{\nu} \leq 2^{\aleph_0} \) for all \( \nu < 2^{\aleph_0} \) then \( |\mathcal{P}(\mathcal{B})/I| = 2^{\aleph_0} \).

PROOF. Part (i) follows immediately from Theorem 2.2(i) and the following result of Comfort-Hager [2] and Monk-Sparks [6]: if \( B \) is an infinite \( \omega_1 \)-complete Boolean algebra, then \( |\mathcal{P}(B)| = |B| \). Since \( |\mathcal{P}(\mathcal{B})/I| \) is \( \omega_1 \)-complete and infinite, this gives us that \( |\mathcal{P}(\mathcal{B})/I| = |\mathcal{P}(\mathcal{B})/I|^\omega \).

Part (ii) is an immediate consequence of Theorem 2.2(ii).

Part (iii) requires the following well-known result of Tarski (see [3, p. 204]): if \( \gamma \) is an infinite cardinal and \( 2^{\alpha} = \gamma \) then there exists a family \( \mathcal{F} \subseteq \gamma \) with \( |\mathcal{F}| = 2^{\gamma} \) and \( |\{\alpha < \gamma : f(\alpha) = g(\alpha)\}| < \gamma \) whenever \( f, g \in \mathcal{F} \) and \( f \neq g \). (To see this, let \( \langle x_\gamma : \beta < \gamma_\alpha \rangle \) be an enumeration of \( \mathcal{P}(\alpha) \) in order-type \( \gamma_\alpha \leq \gamma \) for each \( \alpha < \gamma \) and for each \( X \subseteq \gamma \) define \( f_X : \gamma \rightarrow \gamma \) by \( f_X(\alpha) = \beta \) iff \( X \cap \alpha = X_\beta \).) In the context of (iii), with \( \gamma = 2^{\aleph_0} (= |\mathcal{B}|) \), this gives us that \( |\mathcal{P}(\mathcal{B})/I| > 2^{2^\aleph_0} \), and so (iii) now follows from (i).

The second equality in part (i) of Theorem 2.3 should be compared with Keisler’s result [5] that \( |\prod \mathcal{B}/D| = |\prod \mathcal{B}/D|^\omega \) for any countably incomplete ultrafilter \( D \) on \( \omega \). Part (ii) of Theorem 2.3 eliminates some saturation requirements from an old (unpublished) result of Kunen (recently rediscovered by Pele [7]) and part (iii) does the same for a result of Prikry [8]. The proofs of Kunen, Pele and Prikry all employed Boolean ultrapowers.

3. Some applications. We conclude with a few applications of the results in §2.

APPLICATION 1. We begin with an application that is due as much to Andrzej Pele (see [7]) as to the present author. It answers the following question that we asked in [9, Problem R, p. 51]: Suppose we have countably many \( \omega_1 \)-complete uniform ideals on the real line \( \mathbb{R} \). Can we then find a subset \( X \) of \( \mathbb{R} \) such that neither \( X \) nor \( \mathbb{R} \setminus X \) belongs to any of the ideals in our collection? An affirmative answer to this can be derived as follows. By Theorem 5.4.2 of [9], it suffices to show that if \( I \) is an \( \omega_1 \)-complete uniform ideal on \( \mathbb{R} \) then \( \mathcal{P}(\mathbb{R})/I \) has no countable dense set (in the forcing theoretic sense). Assume we have such an \( I \) and \( \{X_n : n \in \omega\} \subseteq I^+ \). To each distinct \( [Y] \in \mathcal{P}(\mathbb{R})/I \) we associate \( \{n \in \omega : [X_n] \subseteq [Y]\} \). Since \( |\mathcal{P}(\mathbb{R})/I| > 2^{\aleph_0} \) by Theorem 2.3(ii), we get \( [Y_1], [Y_2] \in \mathcal{P}(\mathbb{R})/I \) with \( [Y_1] = [Y_2] \) and \( [X_n] \leq [Y_1] \) iff \( [X_n] \leq [Y_2] \) for every \( n \in \omega \). But this shows that \( \{X_n : n \in \omega\} \) is not a dense set since either \( Y_1 - Y_2 \subseteq I^+ \) or \( Y_2 - Y_1 \subseteq I^+ \) and neither contains \( X_n \) (modulo \( I \)) for any \( n \in \omega \).

APPLICATION 2. Theorem 2.3(ii) also yields yet another proof that not every set of reals is Lebesgue measurable. To see this, assume the contrary and let \( I = \{X \subseteq \mathbb{R} : \mu(X) = 0\} \). Then \( I \) is clearly \( \omega_1 \)-complete and it is uniform since every set of reals of cardinality less than the continuum has outer measure zero. Theorem 2.3(ii) now implies that \( |\mathcal{P}(\mathcal{B})/I| > 2^{\aleph_0} \), and this contradicts the well-known fact that every measurable set differs from a Borel set by a set of measure zero (since there are only \( 2^{\aleph_0} \) Borel sets).

APPLICATION 3. We showed above that a consequence of Theorem 2.3 is the fact that no \( \omega_1 \)-complete uniform ideal \( I \) on \( \mathbb{R} \) can yield a Boolean algebra \( \mathcal{P}(\mathcal{B})/I \)
with a countable dense set. A natural question (again related to versions of Ulam’s problem in [9 and 10]) is whether a dense set of size \( \aleph_1 \) is possible. Woodin [11] has shown that relative to large cardinals an affirmative answer is consistent. In his model CH holds. As a final application of Theorem 2.2 we consider the question of whether CH is necessary and prove the following: if \( \mathcal{R} \) carries an \( \omega_1 \)-complete uniform ideal \( I \) so that \( \mathcal{P}(\mathcal{R})/I \) has a dense set of size \( \aleph_1 \) then either \( 2^{\aleph_0} = \aleph_1 \) or \( 2^{\aleph_0} > \aleph_{\omega_1} \). To see this we argue as follows. Suppose \( I \) is an \( \omega_1 \)-complete, uniform ideal on \( \mathcal{R} \) and \( D \) is a dense set of size \( \aleph_1 \) for \( \mathcal{P}(\mathcal{R})/I \). Define \( \phi: \mathcal{P}(\mathcal{R})/I \rightarrow \mathcal{P}(D) \) by \( \phi([A]) = \{ d \in D : d \leq [A] \} \). It is easy to see that \( \phi \) is one-to-one and so \( |\mathcal{P}(\mathcal{R})/I| \leq 2^{\aleph_1} \). On the other hand, Theorem 2.2(ii) guarantees that \( |\mathcal{P}(\mathcal{R})/I| > 2^{\aleph_0} \), and so \( 2^{\aleph_0} < 2^{\aleph_1} \). Now, if \( \kappa \) is the exact completeness of \( I \) it is easy to see that \( I \) projects to a \( \kappa \)-complete ideal on \( \kappa \), and since \( I \) has a dense set of size of \( \omega_1 \) it is clearly \( \omega_2 \)-saturated. Thus, if \( 2^{\aleph_0} < \aleph_{\omega_1} \) we must have \( \kappa = \omega_1 \) and so we arrive at a situation where we have an \( \omega_2 \)-saturated \( \omega_1 \)-complete ideal on \( \omega_1 \) with \( 2^{\aleph_0} < 2^{\aleph_1} \). A well-known result of Jech-Prikry [4, p. 3] now yields \( 2^{\aleph_0} = \aleph_1 \) as desired.

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