PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 103, Number 1, May 1988

SHORTER NOTES

The purpose of this department is to publish very short papers of unusually polished character, for which there is no other outlet.

ON GAUSS'S FIRST PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA

S. M. GERSTEN AND JOHN R. STALLINGS

(Communicated by Haynes R. Miller)

ABSTRACT. Gauss's first proof of the Fundamental Theorem of Algebra is shown to be related to basic properties of free groups.

While perusing Struik's book [**5**, pp. 115ff.], we observed that Gauss's original proof of the Fundamental Theorem of Algebra, [**1**, pp. 1–31], appeared to bear upon criteria for words in a free group to represent the identity element. Here we explain this observation. It is remarkable that Gauss himself was not able to fill in the details, except to give a plausibility argument at the point in the proof where the free group enters; the deep point in this argument involves the geometry of the 2-cell, especially a version of the Jordan Curve Theorem which is used in the geometric theory of free groups.

Consider the free group F freely generated by elements x and y.

LEMMA 1. *Let*

$$w = x^{\varepsilon_1} y^{\varepsilon_2} y^{\varepsilon_3} \cdots x^{\varepsilon_{4n-1}} y^{\varepsilon_{4n}} \in F,$$

where $\varepsilon_i = \pm 1$ *for all* i. *Then* $w \neq 1$.

This is immediate since the word w is a normal form.

LEMMA 2. *Let n distinct points $P_1, P_2, \ldots, P_n$ be fixed in order around the unit circle in $\mathbf{C}$. Assign to P_i the label λ_i, where λ_i is one of the letters x, x^{-1}, y, or y^{-1}. A necessary and sufficient condition that the word $\lambda_1\lambda_2\cdots\lambda_n$ represent 1 in F is that there exists a family of pairwise disjoint arcs $\mathcal{E}$ properly imbedded in the unit disc in $\mathbf{C}$ with the set of end points of the arcs equal to the set of points $\{P_1, P_2, \ldots, P_n\}$ such that if A is any arc of $\mathcal{E}$ and L is the label of one end point of A, then L^{-1} is the label of the other end point of A.*

This criterion follows from the results of [**2**, §3].

Received by the editors October 22, 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 20E05, 20F32, 12D10.

Key words and phrases. Free group, polynomial.

This work is based on research partly supported by the National Science Foundation under Grants No. DMS-8601376 (Gersten) and DMS-8600320 (Stallings).

©1988 American Mathematical Society
0002-9939/88 $1.00 + $.25 per page

THEOREM. *Let $f(z) = z^n + a_1 z^{n-1} + \cdots + a_n$, with $a_i \in \mathbf{C}$ and $n \geq 1$. Then f has a root in $\mathbf{C}$.*

PROOF. Write $f = g + ih$, where $g, h\colon \mathbf{C} \to \mathbf{R}$ are the real and imaginary parts of f. Gauss's idea was to examine the real algebraic sets $g^{-1}(0)$ and $h^{-1}(0)$ and to claim, by a "geometria situs" argument, that these sets must intersect. Gauss's intuition was correct. Here is how we make this argument precise:

As functions of θ, the expressions $r^{-n}g(re^{i\theta})$ and $r^{-n}h(re^{i\theta})$ converge uniformly, as $r \to +\infty$, to $\cos n\theta$ and $\sin n\theta$, respectively; in addition, the derivatives with respect to θ converge uniformly to the derivatives of these functions. It follows that on a circle of large radius R, the roots of g and of h are each $2n$ in number; the roots of g alternate with the roots of h in order around the circle. Furthermore, this pattern of roots is *stable*: There is $\delta > 0$ such that, if ε is any constant with $|\varepsilon| < \delta$, then the roots of $g - \varepsilon$ and of $h - \varepsilon$ exhibit the same pattern.

Now, assume that for all $z \in \mathbf{C}$, $f(z) \neq 0$. Thus, $g^{-1}(0) \cap h^{-1}(0) = \varnothing$. Let $M = \{z \in \mathbf{C}\colon |z| \leq R\}$, and restrict g and h to M. By Sard's Theorem, [**4**, p. 47], the critical values of $g|M$ and $h|M$ are both of measure zero. Thus we may choose $\varepsilon > 0$, sufficiently small so that

(1) $g^{-1}(\varepsilon)$ and $h^{-1}(\varepsilon)$ are 1-manifolds properly imbedded in M.

(2) $g^{-1}(\varepsilon) \cap h^{-1}(\varepsilon) = \varnothing$.

(3) The set $g^{-1}(\varepsilon) \cap \partial M$ is close to $g^{-1}(0) \cap \partial M$, and similarly the set $h^{-1}(\varepsilon) \cap \partial M$ is close to $h^{-1}(0) \cap \partial M$.

Now for each arc component of $g^{-1}(\varepsilon)$, label one end point x and the other one x^{-1}. For each arc component of $h^{-1}(\varepsilon)$, label one end point y and the other one y^{-1}. The resulting cyclic word around ∂M is one of the form

$$w = x^{\varepsilon_1} y^{\varepsilon_2} x^{\varepsilon_3} \cdots x^{\varepsilon_{4n-1}} y^{\varepsilon_{4n}}.$$

Hence $w \neq 1$ by Lemma 1. But the arcs of the collections $g^{-1}(\varepsilon)$ and $h^{-1}(\varepsilon)$ satisfy the hypotheses of Lemma 2. Thus $w = 1$. This contradiction establishes the theorem.

REMARK. A modern guide to literature on the Fundamental Theorem of Algebra can be found in [**3**].

REFERENCES

1. C. F. Gauss, *Werke*, vol. 3, Georg Olms Verlag, Hildesheim and New York, 1973.
2. R. Z. Goldstein and E. C. Turner, *Applications of topological graph theory to group theory*, Math. Z. **165** (1979), 1 10.
3. S. Smale, *The Fundamental Theorem of Algebra and complexity theory*, Bull. Amer. Math. Soc. (N.S.) **4** (1981), 1 36.
4. S. Sternberg, *Lectures on differential geometry*, Prentice-Hall, 1964.
5. D. J. Struik, *A source book in mathematics* 1200 1800, Harvard Univ. Press, 1969.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH, SALT LAKE CITY, UTAH 84112

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720