

INJECTIVE DIMENSION OF SOME DIVISIBLE MODULES OVER A VALUATION DOMAIN

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ABSTRACT. Let R be a valuation domain of global dimension $n + 1$. Given an infinite direct product of injective envelopes of (torsion) cyclic modules, let D_{n-k} be the submodule consisting of the elements having support of cardinality less than \aleph_{n-k} .

We prove that the injective dimension of D_{n-k} is at most k and, using \circ -axiom, we prove that D_{n-2} has injective dimension exactly 2.

Introduction. Given a valuation domain R it is well known that, if every ideal of R is countably generated, then every divisible R -module is either injective or of injective dimension 1. (See [FS, Chapter 6].) Very little seems to be known if R admits ideals of uncountable type or, equivalently if global dimension $R \geq 3$.

Here we address this question for R -modules of a particular type. Let global dimension $R = n + 1$ (≥ 3); given an infinite direct product of injective envelopes of cyclics R/L_α (L_α archimedean ideals) consider the submodule D_{n-k} ($0 \leq k \leq n$) of the product consisting of all the elements having support of cardinality $< \aleph_{n-k}$. We prove that the injective dimension of D_{n-k} is at most k , while D_{n-1} has injective dimension exactly 1.

The case $k = 2$ is more difficult. Using the combinatorial principle known as \circ -axiom we prove that D_{n-2} has injective dimension 2.

If global dimension $R = 3$, the module D_{n-2} is a direct sum of injectives and we do not know if it is consistent with ZFC to assume that its injective dimension is 1.

We are indebted to Professor L. Fuchs for proposing to us the question discussed in this paper and for suggesting to us the use of \circ -axiom for its solution.

1. The modules D_{n-k} . R will always denote a valuation domain, P its maximal ideal and Q its quotient field.

If J is an ideal of R , $\text{gen } J$ will denote the least cardinality of generating systems for J and we will say that J is κ -generated (for a cardinal κ) in case $\text{gen } J \leq \kappa$. (\aleph_{-1} means finitely generated.)

If M is an R -module and J an ideal of R , $M[J]$ will denote the submodule of M consisting of the elements annihilated by J . Throughout we will use well-known results about the projective dimension of modules over R (see [O]).

Let global dimension $R = n + 1$. Baer's criterion says that an R -module M is injective if and only if $\text{Ext}^1(R/J, M) = 0$ for every \aleph_{n-1} -generated ideal J .

The object of the present paper is to investigate the following problem.

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PROBLEM 1. Suppose M is an R -module satisfying $\text{Ext}^1(R/J, M) = 0$ for every \aleph_m -generated ideal J , for some $m \in N$. What can be said about the injective dimension of M ?

The first result we can establish is the following.

PROPOSITION 1. *Let global dimension $R = n + 1$ and M an R -module such that $\text{Ext}^1(R/J, M) = 0$ for every \aleph_{n-k-1} -generated ideal J ($0 \leq k \leq n$). Then the injective dimension of M is at most k .*

PROOF (The case $k = 0$ is just Baer's criterion). We first prove that (*) $\text{Ext}^1(N, M) = 0$ for every R -module N of projective dimension at most $n - k + 1$.

In fact, by Corollary 4 in [BF], we have $N = \bigcup_{\alpha < \tau} N_\alpha$ where $\{N_\alpha\}_{\alpha < \tau}$ is a smooth chain of submodules of N such that $N_{\alpha+1}/N_\alpha \approx R/J$ with J \aleph_{n-k-1} -generated. A simple application of Lemma 2.1, Chapter IV [FS] yields (*). Using induction on m we can now prove $\text{Ext}^{m+1}(N, M) = 0$ for every R -module N of projective dimension at most $n - k + 1 + m$.

The case $m = 0$ is just (*).

Let N be of projective dimension at most $n - k + 1 + m$ and consider an exact sequence: (1) $0 \rightarrow N_1 \rightarrow F \rightarrow N \rightarrow 0$ where F is a free module; then projective dimension $N_1 \leq n - k + m$ and by induction $\text{Ext}^m(N, M) = 0$. But the exact sequence (1) yields $\text{Ext}^m(N_1, M) \approx \text{Ext}^{m+1}(N, M)$. \square

This natural question now arises:

QUESTION 1. Let R and M be as in Proposition 1. Assume there exists an ideal J such that $\text{gen } J = \aleph_{n-k}$ and $\text{Ext}^1(R/J, M) \neq 0$. Can we prove that the injective dimension of M is exactly k ?

The hypothesis $\text{Ext}^1(R/J, M) = 0$ for every \aleph_{n-k-1} -generated ideal J , implies $\text{Ext}^1(R/J, M) = 0$ for every finitely generated ideal J (since $k \geq 0$) and this is equivalent to saying that M is divisible. So from now on we will assume M is divisible and moreover we will concentrate on a special class of divisible modules.

We now set down some notations which will be frequently used in the sequel.

Let $\{L_\alpha\}_{\alpha \in \Lambda}$ be a family of archimedean ideals of R (i.e. $rL_\alpha \not\subseteq L_\alpha$, for every nonunit $r \in R$) and let I_α be the injective envelope of R/L_α , $\alpha \in \Lambda$. (Note that $I_\alpha \geq Q/L_\alpha$.) For $n \in N$ and $0 \leq k \leq n$ denote by D_{n-k} the submodule of $\prod_{\alpha \in \Lambda} I_\alpha$ consisting of all the elements whose supports have cardinality less than \aleph_{n-k} . We obviously assume Λ is big enough, namely $|\Lambda| \geq \aleph_n$. The aim of the following two propositions is to establish that the module D_{n-k} satisfies the requirements mentioned in Question 1.

For every ideal J of R we need to consider a filtration. Assume $\text{gen } J = \aleph_k$; we construct, by transfinite induction, a smooth chain $\{J_\lambda\}_{\lambda < \aleph_k}$ of ideals in J as follows.

J_1 is generated by an element j_1 in J .

If λ is a limit ordinal ($\lambda < \aleph_k$), $J_\lambda = \bigcup_{\nu < \lambda} J_\nu$.

If ν is nonlimit, $\nu < \aleph_k$, J_ν is generated by an element j_ν in J satisfying: (i) if $\nu = \mu + 1$, μ nonlimit, j_ν is an element not in J_μ . (ii) If $\nu = \lambda + 1$, λ limit, j_ν is an element not in $J_\lambda (= \bigcup_{\mu < \lambda} J_\mu)$ with $\langle j_\nu \rangle / J_\lambda$ not isomorphic to R/P in case P is not principal.

Next we want $J = \bigcup_{\lambda < \aleph_k} J_\lambda$. It is easy to see that such a filtration always exists for every ideal J . We will call it a regular filtration. The point to be checked is

the construction of the element j_ν , $\nu = \lambda + 1$, λ limit; satisfying condition (ii). Consider $a \in J \setminus J_\lambda$; assume P is not principal and $\langle a \rangle / J_\lambda$ isomorphic to R/P . This means $R/a^{-1}J_\lambda$ isomorphic to R/P or, equivalently, $a^{-1}J_\lambda = P$. Let b be an element in $J \setminus \langle a \rangle$; there is $r \in P$ such that $rb = a$ and thus $\langle b \rangle / J_\lambda$ cannot be isomorphic to R/P since $b^{-1}J_\lambda = ra^{-1}J_\lambda = rP$ which is strictly contained in P . Put then $b = j_\nu$.

PROPOSITION 2. $\text{Ext}^1(R/J, D_{n-k}) = 0$ for every ideal J such that $\text{gen } J \neq \aleph_{n-k}$.

PROOF. We prove that if J is as stated and $f: J \rightarrow D_{n-k}$ is a homomorphism, then f is extendable to R . Let $\{J_\lambda\}_{\lambda < \text{gen } J}$ be a regular filtration for J . Define the support of f as the union of the supports of the elements $f(j_\nu)$, for ν nonlimit, $\nu < \text{gen } J$.

Case 1. $\text{gen } J \not\leq \aleph_{n-k}$. It is clear that in this case $|\text{supp } f| \leq \aleph_{n-k}$; thus $f(J) \leq \prod_{\alpha \in \Lambda_1} I_\alpha$ where $\Lambda_1 = \text{supp } f$, and f is extendable to R .

Case 2. $\text{gen } J \not\geq \aleph_{n-k}$. As in the first case it suffices to prove that $|\text{supp } f| \not\geq \aleph_{n-k}$. By way of contradiction suppose $|\text{supp } f| \geq \aleph_{n-k}$ and choose J_0 in J , J_0 \aleph_{n-k} -generated with $|\text{supp } f|_{J_0} = \aleph_{n-k}$. ($f|_{J_0}$ denotes the restriction of f to J_0 .) Take $j \in J \setminus J_0$, $jR \not\supseteq J_0$; hence $\text{supp } f(j) \supseteq \text{supp } f|_{J_0}$ which is absurd since $\text{supp } f(j)$ has to be of cardinality less than \aleph_{n-k} . \square

PROPOSITION 3. $\text{Ext}^1(R/J, D_{n-k}) \neq 0$ for every ideal J with $\text{gen } J = \aleph_{n-k}$. Moreover given an increasing chain $\{X_\nu: \nu \text{ nonlimit}, \nu < \aleph_{n-k}\}$ of subsets of Λ each one of cardinality less than \aleph_{n-k} , and a regular filtration $\{J_\lambda\}_{\lambda < \aleph_{n-k}}$, there is a homomorphism $f: J \rightarrow D_{n-k}$ such that $\text{supp } f|_{J_\nu} = X_\nu$, for every nonlimit $\nu < \aleph_{n-k}$.

PROOF. It is enough to prove the second part since, if we choose the chain $\{X_\nu\}$ satisfying $|\bigcup_\nu X_\nu| = \aleph_{n-k}$, then the stated homomorphism f is not extendable to R .

Let μ, ν denote nonlimit ordinals, $\mu < \nu < \aleph_{n-k}$. We have $r_\mu^\nu j_\nu = j_\mu$ for r_μ^ν non units in R and j_μ, j_ν the generators of the corresponding elements J_μ, J_ν in the regular filtration.

By transfinite induction define elements d_ν in D_{n-k} such that (i) $r_\mu^\nu d_\nu = d_\mu$, (ii) $\text{supp } d_\nu = X_\nu$. Then $f(j_\nu) = d_\nu$ will yield the required homomorphism. Let d_1 be any element of D_{n-k} with support X_1 . Suppose d_μ has been chosen for every $\mu < \nu$. Define d_ν as follows.

Case 1. $\nu = \mu + 1$, μ nonlimit. If $\alpha \in X_\mu$, let $d_\nu(\alpha)$ be a nonzero element in I_α such that $r_\mu^\nu d_\nu(\alpha) = d_\mu(\alpha)$. If $\alpha \in X_\nu \setminus X_\mu$ let $d_\nu(\alpha)$ be $q + L_\alpha$ with q in $(r_\mu^\nu)^{-1}L_\alpha \setminus L_\alpha$; there exists such a q by the hypothesis L_α archimedean. Obviously $d_\nu(\alpha) = 0$ if $\alpha \notin X_\nu$.

Case 2. $\nu = \lambda + 1$, λ limit. Let $\Lambda_0 = \bigcup_\mu \{X_\mu: \mu < \nu\}$. Clearly $|\Lambda_0| \not\geq \aleph_{n-k}$ and the elements d_μ ($\mu < \nu$) already constructed lie in $\prod_{\alpha \in \Lambda_0} I_\alpha = I_0$. The system of equations $r_\mu^\nu x = d_\mu$ is finitely solvable in I_0 and hence it has a solution d in I_0 ; put $d_\nu(\alpha) = d(\alpha)$ for $\alpha \in \Lambda_0$. For $\alpha \in X_\nu \setminus \Lambda_0$ let $d_\nu(\alpha) = q + L_\alpha$ with q an element of $\bigcap_\mu \{(r_\mu^\nu)^{-1}L_\alpha \setminus L_\alpha, \mu < \nu, \mu \text{ nonlimit}\}$. The intersection is not empty by property (ii) of a regular filtration.

Let $d_\nu(\alpha) = 0$ for $\alpha \notin X_\nu$. Obviously the family $\{d_\nu: \nu < \aleph_{n-k}, \nu \text{ nonlimit}\}$ satisfies (i) and (ii) above and thus the conclusion follows. \square

2. The injective dimension of D_{n-2} . In this section we inquire about the injective dimension of the modules D_{n-k} .

Assume global dimension $R = n + 1$. We have:

FACT 1. Injective dimension $D_{n-k} \leq k$.

Obvious from Propositions 1 and 2. Thus, in particular, injective dimension $D_n = 0$.

FACT 2. Injective dimension $D_{n-1} = 1$.

In fact D_{n-1} is not injective by Proposition 3. Thus the interesting case is for $n \geq 2$, hence we will assume global dimension $R \geq 3$.

Now, by Fact 1 and Proposition 3, we infer that the injective dimension of D_{n-2} is 1 or 2.

The exact value does not seem decidable in ZFC. So from now on we will concentrate on D_{n-2} . First of all, notice that, if Λ_1 is a subset of Λ of cardinality \aleph_{n-2} , then the elements of D_{n-2} with supports in Λ_1 form a direct summand of D_{n-2} . Thus if we want to prove that injective dimension $D_{n-2} = 2$ we can assume $|\Lambda| = \aleph_{n-2}$. Consider the exact sequence

$$(1) \quad 0 \rightarrow D_{n-2} \rightarrow I \rightarrow C \rightarrow 0 \quad \text{where } I = \prod_{\alpha \in \Lambda} I_\alpha, \quad |\Lambda| = \aleph_{n-2}.$$

By the previous remark, the injective dimension of C is 0 or 1 and clearly C is injective if and only if every homomorphism from every J to C has a lifting to I .

LEMMA 1. *Let J be \aleph_{n-2} -generated and $\{J_\lambda\}_{\lambda < \text{gen } J}$ a regular filtration for J . Then*

- (i) $\text{Ext}^1(J, D_{n-2}) = 0$.
- (ii) $\text{Hom}(J, I) \rightarrow \text{Hom}(J, C) \rightarrow 0$ is exact.
- (iii) *Given a family $\{y_\nu: \nu \text{ nonlimit}, \nu < \text{gen } J\}$ of elements of I with*
 - (a) $j_\mu(y_\nu - y_\mu) \in D_{n-2}$ for every $\mu < \nu < \text{gen } J$ (μ nonlimit), *there exists an element $y \in I$ such that*
 - (b) $j_\nu(y - y_\nu) \in D_{n-2}$ for every $\nu < \text{gen } J$.

PROOF. (i) Write $J = \bigcup_{\lambda < \text{gen } J} J_\lambda$. Then $J_{\lambda+1}/J_\lambda$ is isomorphic to R/J' with $\text{gen } J' \not\leq \aleph_{n-2}$. Thus, by Proposition 2, $\text{Ext}^1(J_{\lambda+1}/J_\lambda, D_{n-2}) = 0$ and by Lemma 2.1, Chapter 2 of [FS] the conclusion follows. \square

(ii) Obvious from (i) and the exact sequence (1).

(iii) is just a reformulation of (ii) since condition (a) guarantees that $f(j_\nu) = j_\nu(y_\nu + D_{n-2})$ is a homomorphism $f: J \rightarrow C$ and f lifts to I exactly if there exists $y \in I$ satisfying condition (b). \square

The following technical lemma is crucial.

LEMMA 2. *Let J be an ideal, $\text{gen } J = \aleph_{n-2}$ and $\{J_\lambda\}_{\lambda < \aleph_{n-2}}$ a regular filtration for J . Let $\{y_\nu: \nu \text{ nonlimit}, \nu < \aleph_{n-2}\}$ be a family of elements of I satisfying condition (a) of Lemma 1(iii). Given a family $\{H_\nu: \nu \text{ nonlimit}, \nu < \aleph_{n-2}\}$ of \aleph_{n-2} subsets of Λ of cardinality less than \aleph_{n-2} , there exist $y \in I$ and a subset A of Λ , $A = \{\alpha_\nu: \nu < \aleph_{n-2}, \nu \text{ nonlimit}\}$ such that*

- (i) $|A| = \aleph_{n-2}$.

(ii) For every ν , $\alpha_\nu \notin H_\nu$; $\text{supp } j_\nu(y - y_\nu)$ has cardinality less than \aleph_{n-2} and contains α_ν .

PROOF. By Lemma 1(iii) there exists $y_0 \in I$ satisfying (b), namely $j_\nu(y_0 - y_\nu) \in D_{n-2}$, for every ν and clearly any other element of I satisfying (b) is in the coset $(y_0 + D_{n-2}) + C[J]$. We want to find an element $z \in I$, with $z + D_{n-2} \in C[J]$, in order that $y_0 + z$ will be the required element. Let μ, ν be nonlimit ordinals $< \aleph_{n-2}$. Put $G_\nu = \text{supp } j_\nu(y_0 - y_\nu)$ and define A inductively as follows: α_1 is an element in $\Lambda \setminus (G_1 \cup H_1)$ and if α_μ has been chosen for every $\mu < \nu$, let α_ν be an element in $\Lambda \setminus (G_\nu \cup H_\nu \cup \{\alpha_\mu : \mu < \nu\})$. Obviously $|A| = \aleph_{n-2}$. For every ν , let $F_\nu = \{\alpha_\mu : \mu \leq \nu\}$. We can apply Proposition 3 to the family $\{F_\nu : \nu < \aleph_{n-2}\}$ to get an element $z \in I$ with $\text{supp } j_\nu z = F_\nu$, for every ν . Take $y = y_0 + z$. Then for every ν , $j_\nu(y - y_\nu) = j_\nu z + j_\nu(y_0 - y_\nu)$. Consequently, $\text{supp } j_\nu(y - y_\nu) \subseteq F_\nu \cup G_\nu$ and since $\alpha_\nu \in F_\nu \setminus G_\nu$, we get $\alpha_\nu \in \text{supp } j_\nu(y - y_\nu)$. \square

Up to now, we know that ever homomorphism from an \aleph_{n-2} -generated ideal J into C lifts to I (Lemma 1). But, since global dimension $R = n + 1$, there are ideals J with $\text{gen } J = \aleph_{n-1}$, and to be able to find homomorphisms from such ideals to C not liftable to I , we assume the axiom $\diamond_{\aleph_{n-1}}$ (due to Jensen) which we are going to use in the following form (see [E]).

Let X be a set $|X| \leq \aleph_{n-1}$, $X = \bigcup_{\lambda < \aleph_{n-1}} X_\lambda$ (smooth filtration), there is a family of functions (\diamond -sequence), $\{h_\lambda : \lambda \rightarrow X_\lambda | \lambda < \aleph_{n-1}\}$ such that, for any function $h : \aleph_{n-1} \rightarrow X$, the set $\{\lambda < \aleph_{n-1} | h|_\lambda = h_\lambda\}$ is stationary.

THEOREM 1. Assume $\diamond_{\aleph_{n-1}} + (2^{\aleph_{n-3}} \leq \aleph_{n-1}, \text{ if } n \geq 3)$. Then the injective dimension of D_{n-2} is 2.

PROOF. We show that for every ideal J with $\text{gen } J = \aleph_{n-1}$, there exists a homomorphism $f : J \rightarrow C$ not liftable to I ; this will imply injective dimension $D_{n-2} = 2$.

Let $\{J_\lambda\}_{\lambda < \aleph_{n-1}}$ be a regular filtration for J . We define inductively the restrictions $f|_{J_\lambda}$ of f to J_λ . They have to be multiplications by elements $y_\lambda + D_{n-2}$, $y_\lambda \in I$, since $\text{gen } J_\lambda < \aleph_{n-1}$ for every $\lambda < \aleph_{n-1}$ (apply Lemma 1). Obviously the elements y_λ satisfy

$$(*) \quad j_\mu(y_\lambda - y_\mu) \in D_{n-2} \quad \text{for every } \mu < \lambda < \aleph_{n-1}, \mu \text{ nonlimit.}$$

Let X be the set of all subsets of Λ of cardinality less than \aleph_{n-2} . Then $|X| = 2^{\aleph_{n-3}} \leq \aleph_{n-1}$ by assumption (in case $n = 2$, $|X| = \aleph_0$ without any hypothesis). Write $X = \bigcup_{\lambda < \aleph_{n-1}} X_\lambda$ (smooth filtration) and consider a $\diamond_{\aleph_{n-1}}$ -sequence $h_\lambda : \lambda \rightarrow X_\lambda$, $\lambda < \aleph_{n-1}$. We define, by induction, a family $\{y_\lambda \in I : \lambda < \aleph_{n-1}\}$ satisfying (*).

- (i) $\lambda = 1$, y_1 is any element in I .
- (ii) λ limit or $\lambda = \rho + 1$ for ρ limit such that $|\rho| \not\leq \aleph_{n-2}$.

Consider the already constructed set $\{y_\mu : \mu \text{ nonlimit}, \mu < \lambda\}$ satisfying (*). Let y_λ be an element satisfying condition (b) of Lemma 1(iii).

- (iii) $\lambda = \nu + 1$, ν nonlimit. Let y_λ be such that $j_\nu(y_\lambda - y_\nu) \in D_{n-2}$.
- (iv) $\lambda = \rho + 1$, ρ limit, $|\rho| = \aleph_{n-2}$.

Consider the function $h_\rho: \rho \rightarrow X_\rho$ and put $h_\rho(\mu) = H_\mu$ for every $\mu < \rho$ μ nonlimit. The family $\{H_\mu: \mu < \rho\}$ satisfies the hypothesis of Lemma 2, therefore we can find an element $y \in I$ such that, if $S_\mu = \text{supp } j_\mu(y - y_\mu)$, S_μ contains an element $\alpha_\mu \in \Lambda \setminus H_\mu$ and $|\{\alpha_\mu: \mu < \rho\}| = \aleph_{n-2}$. Let y_λ be such a y . It is clear that the set $\{y_\lambda: \lambda < \aleph_{n-1}\}$ constructed above satisfies (*) and thus $f = \bigcup_\lambda \{f|_{J_\lambda}: \lambda < \aleph_{n-1}\}$ gives rise to a homomorphism $f: J \rightarrow C$.

CLAIM. f is not liftable to I .

Suppose the contrary. This means there is a family $\{d_\lambda: d_\lambda \in D_{n-2}, \lambda < \aleph_{n-1}\}$ satisfying (**) $j_\mu(y_\lambda + d_\lambda - (y_\mu + d_\mu)) = 0$ for every $\mu < \lambda < \aleph_{n-1}$, μ nonlimit. Consider the function $h: \aleph_{n-1} \rightarrow X$, $h(\lambda) = \text{supp } d_\lambda = F_\lambda$, for every $\lambda < \aleph_{n-1}$. By $\diamond_{\aleph_{n-1}}$, there is a limit ordinal ρ_0 with $|\rho_0| = \aleph_{n-2}$ (the set of such ρ_0 's is stationary), such that $h(\rho_0) = h|_{\rho_0}$, i.e. $H_\mu = F_\mu$ for every $\mu < \rho_0$, μ nonlimit.

Consider the element y_{ρ_0+1} constructed in (iv). We have $j_\mu(y_{\rho_0+1} - y_\mu) = -j_\mu(d_{\rho_0+1} - d_\mu)$, for every $\mu < \rho_0$, μ nonlimit; hence, $S_\mu = \text{supp } j_\mu(y_{\rho_0+1} - y_\mu) \subseteq F_{\rho_0+1} \cup F_\mu$. but $\alpha_\mu \in S_\mu \setminus F_\mu$, implies $\alpha_\mu \in F_{\rho_0+1}$, for every $\mu < \rho_0$, contradicting $|F_{\rho_0+1}| \not\leq \aleph_{n-2}$. \square

REMARK. Let global dimension $R = 3$. D_{n-2} is just $\bigoplus_{n < \omega} I_n$ and Theorem 1 says that \diamond_{\aleph_1} implies injective dimension $\bigoplus_{n < \omega} I_n = 2$.

We end with this natural question.

Is there an axiom which implies that injective dimension $\bigoplus_{n < \omega} I_n = 1$?

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