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ON THE FINITENESS OF THE PROJECTIVE DIMENSION OF CERTAIN IDEALS

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ABSTRACT. For a local noetherian ring, a subset of a minimal system of generators of the maximal ideal is a regular sequence if the ideal it generates has finite projective dimension.

We prove

THEOREM. *Let (A, m, K) be a local noetherian ring and $\{x_1, \ldots, x_n\}$ a minimal set of generators for m. If the ideal of A generated by $x_1, \ldots, x_r$ $(r \leq n)$ has finite projective dimension, then $x_1, \ldots, x_r$ is a regular sequence.*

For $r = n$ we obtain the Serre's theorem: a local noetherian ring of finite global dimension is regular. We also obtain the well-known result: each subset of a regular system of parameters of a regular local ring is a regular sequence.

To prove the theorem we use

LEMMA. *Let (A, m, K) be a local noetherian ring and x a nonzero divisor in A which is not contained in m^2. Let N be a finitely generated $A/(x)$-module. If N has finite projective dimension as an A-module, then N has finite projective as an $A/(x)$-module.*

PROOF. Since each free $A/(x)$-module has projective dimension one as an A-module, it suffices to consider that N has projective dimension one as an A-module. Let $0 \to R \to F \to N \to 0$ be an exact sequence with $R \subset mF$, R and F being free A-modules. Let $\{b_1, \ldots, b_n\}$ be a basis of F. The elements $xb_1, \ldots, xb_n$ are in R and their classes $\operatorname{mod} mR$ are k-linearly independent. Hence $\{xb_1, \ldots, xb_n\}$ is a basis of R, since the rank of R cannot be larger than the rank of F. Therefore N is a free $A/(x)$-module.

We give now an alternative proof of the lemma, which is more complicated, but it provides a formula for relating the Betti numbers of N relative to A and $A/(x)$. We shall show that there exists an isomorphism

$$\operatorname{Tor}_{n+1}^{A}(N, K) \simeq \operatorname{Tor}_{n+1}^{A/(x)}(N, K) \oplus \operatorname{Tor}_{n}^{A/(x)}(N, K) \quad \text{for } n \geq 1.$$

We consider the change-rings spectral sequence

$$E_{p,q}^2 = \operatorname{Tor}_p^{A/(x)}(N, \operatorname{Tor}_q^A(A/(x), m)) \Rightarrow \operatorname{Tor}_{p+q}^A(N, m).$$

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Since $E^2_{p,0} = \operatorname{Tor}_p^{A/(x)}(N, m/xm)$ and $E^2_{p,q} = 0$ for $q > 0$, the spectral sequence degenerates and we have isomorphisms $\operatorname{Tor}_n^{A/(x)}(N, m/xm) \simeq \operatorname{Tor}_n^A(N, m)$ for $n \geq 1$. On the other hand there exists an isomorphism of $A/(x)$-modules $m/xm \simeq m/(x) \oplus K$ [**2**, Lemma 2]. Hence $\operatorname{Tor}_n^{A/(x)}(N, m/(x)) \oplus \operatorname{Tor}_n^A(N, K) \simeq \operatorname{Tor}_n^A(N, m)$ for $n \geq 1$ and the result follows.

PROOF OF THE THEOREM. Let I be the ideal of A generated by $x_1, \ldots, x_r$. It suffices to show that I is generated by a regular sequence [**1**, Proposition 1.4.8]. Induction on r. Since I has finite projective dimension, there exists a nonzero divisor in $I - mI$ [**2**, Lemma 3]. Choosing, if necessary, a new system of generators of m, we may assume that x_1 is a nonzero divisor. Therefore the lemma implies that $I/(x_1)$ has finite projective dimension as an $A/(x_1)$-module. By induction $x_2, \ldots, x_r$ is a regular sequence $\operatorname{mod}(x_1)$. Then $x_1, \ldots, x_r$ is a regular sequence.

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REFERENCES

1. T. H. Gulliksen and G. Levin, *Homology of local rings*, Queen's Papers in Pure and Appl. Math., no. 20, Queen's Univ., Kingston, Canada, 1969
2. W. V. Vasconcelos, *Ideals generated by R-sequences*, J. Algebra **6** (1967), 309–316.

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