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AN INEQUALITY FOR THE INTEGRAL MEANS OF A HADAMARD PRODUCT

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ABSTRACT. Motivated by Colzani's paper [1] we prove that

$$M_q(r, f * g) \leq (1-r)^{1-1/p}\|f\|_p\|g\|_q, \qquad 0 < r < 1,$$

where $0 < p < 1$, $p \leq q \leq \infty$ and $f * g$ is the Hadamard product of $f \in H^p$ and $g \in H^q$.

For a function F, continuous in the disc $U = \{z : |z| < 1\}$ let

$$M_p^p(r, F) = \frac{1}{2\pi}\int_0^{2\pi} |F(re^{it})|^p \, dt, \qquad 0 < p < \infty,$$
$$M_\infty(r, F) = \max_{0 \leq t < 2\pi} |F(re^{it})|,$$

where $0 \leq r < 1$. The Hardy class H^p consists of those f which are analytic in U and satisfy the condition

$$\|f\|_p := \sup_{0<r<1} M_p(r, f) < \infty.$$

If $f(z) = \sum a_n z^n$ and $g(z) = \sum b_n z^n$ are analtyic in U, then their Hadamard product $f * g$, defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n,$$

is analytic in U. It is well known that if $f \in H^1$ and $g \in H^q$, $q \geq 1$, then $M_q(r, f * g) \leq \|f\|_1\|g\|_q$ $(0 < r < 1)$ and consequently $f * g \in H^q$. This fact is generalized by the following theorem.

THEOREM. *Let $f \in H^p$ and $g \in H^q$, where $0 < p \leq 1$ and $p \leq q < \infty$. Then*

$$M_q(r, f * g) = O((1-r)^{1-1/p}), \qquad r \to 1^-,$$

and,

$$M_q(r, f * g) \leq (1-r)^{1-1/p}\|f\|_p\|g\|_q, \qquad 0 < r < 1. \tag{1}$$

For the proof we need a familiar lemma.

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LEMMA. *If $F \in H^p$, $0 < p \leq 1$, then*

$$M_1(r, F) \leq (1 - r^2)^{1-1/p} \|F\|_p, \qquad 0 < r < 1.$$

PROOF. Since

$$M_1(r, F) \leq M_\infty^{1-p}(r, F) M_p^p(r, F),$$

we have to prove that

$$M_\infty(r, F) \leq (1 - r^2)^{-1/p} \|F\|_p.$$

This is reduced to the case $p = 2$ in the standard way (see [**2**]). Let $F(z) = \sum c_n z^n$ belong to H^2. Then

$$\begin{aligned} M_\infty^2(r, F) &\leq \left(\sum_0^\infty |c_n| r^n \right)^2 \\ &\leq \sum_0^\infty |c_n|^2 \sum_0^\infty r^{2n} = \|F\|_2^2 (1 - r^2)^{-1}, \end{aligned}$$

and this concludes the proof.

PROOF OF THE THEOREM. It is easily shown that f and g may be supposed to be analytic in the closed disc. By Parseval's formula

$$h(r^2 w) := (f * g)(r^2 w) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{-it}) g(re^{it} w)\, dt, \qquad |w| = 1,$$

whence

$$\begin{aligned} |h(r^2 w)| &\leq \frac{1}{2\pi} \int_0^{2\pi} |\overline{f(re^{-it})}| |g(re^{it} w)|\, dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} |F(re^{it})|\, dt = M_1(r, F), \end{aligned}$$

where

$$F(z) = \overline{f(\overline{z})} g(zw), \qquad |z| \leq 1.$$

Since f is analytic in the closed unit disc we may apply the lemma to obtain

$$\begin{aligned} (1 - r^2)^{1-p} |h(r^2 w)|^p &\leq \|F\|_p^p \\ &= \frac{1}{2\pi} \int_0^{2\pi} |f(e^{-it})|^p |g(e^{it} w)|^p\, dt. \end{aligned}$$

Hence, by Minkowski's inequality (in continuous form),

$$\begin{aligned} &(1 - r^2)^{1-p} M_s(r^2, |h|^p) \\ &\quad \leq \frac{1}{2\pi} \int_0^{2\pi} |f(e^{-it})|^p\, dt \left\{ \frac{1}{2\pi} \int_0^{2\pi} |g(e^{it} e^{i\theta})|^{ps}\, d\theta \right\}^{1/s}, \end{aligned}$$

where $s \geq 1$. By taking $s = q/p$ we get

$$(1 - r^2)^{1-p} M_q^p(r^2, h) = (1 - r^2)^{1-p} M_s(r^2, |h|^p) \leq \|f\|_p^p \|g\|_q^p,$$

and this concludes the proof.

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