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SUPPORT POINTS OF SUBORDINATION FAMILIES

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ABSTRACT. Let $s(F)$ denote the set of functions subordinate to a function F analytic in the unit disc Δ. Let $Hs(F)$ denote the closed convex hull of $s(F)$ and $\operatorname{supp} s(F)$ the set of support points of $s(F)$. We prove the following

THEOREM. *Let F be analytic in Δ and satisfy*

(1) $Hs(F) = \{\int_{\partial\Delta} F(xz)d\mu(x)\colon \mu$ *a probablity measure on* $\partial\Delta\}$ *and*

(2) $F(z) = G(z)/(z - x_0)^\alpha$ *where G is analytic in Δ, continuous in $\overline{\Delta}$, $G(x_0) \neq 0$ and $\alpha > 1$. Then* $\operatorname{supp} s(F) = \{F(xz)\colon |x| = 1\}$.

Introduction. Let $\Delta = \{z\colon |z| < 1\}$ and let $\mathbf{A}$ denote the set of functions analytic in Δ. Let $B_0 = \{\phi\colon \phi \in \mathbf{A}$ and $|\phi(z)| \leq |z|\}$. $\mathbf{A}$ is a locally convex linear topological space with respect to the topology given by uniform convergence on compact subsets of Δ. By a continuous, linear functional on $\mathbf{A}$, we mean a complex-valued functional defined on $\mathbf{A}$ that is continuous and linear. In other words, if J is such a functional, then $J(af + bg) = aJ(f) + bJ(g)$ whenever a and b belong to $\mathbf{C}$ and f and g belong to $\mathbf{A}$. Also, if $f_n \in \mathbf{A}$ $(n = 1, 2, \dots)$ and $f_n \to f$, then $J(f_n) \to J(f)$. Each continuous, linear functional J on $\mathbf{A}$ is given by a sequence $\{b_n\}$ $(n = 1, 2, \dots)$ which satisfies $\overline{\lim}_{n\to\infty} \sqrt{|b_n|} < 1$ and

$$J(f) = \sum_{n=0}^{\infty} a_n b_n \tag{1}$$

where $f(z) = \sum_{n=0} a_n z^n$ $(|z| < 1)$ [7]. For such a sequence $\{b_n\}$, the function $F(z) = \sum_{n=0}^{\infty} b_n z^n$ is analytic in $\overline{\Delta} = \{z\colon |z| \leq 1\}$.

A function f is called a support point of a compact subset $\mathbf{F}$ of $\mathbf{A}$ if $f \in \mathbf{F}$ and if there is a continuous, linear functional J on $\mathbf{A}$ so that

$$\operatorname{Re} J(f) = \max\{\operatorname{Re} J(g)\colon g \in \mathbf{F}\} \tag{2}$$

and $\operatorname{Re} J$ is not constant on $\mathbf{F}$. In general, for a fixed continuous, linear functional J, the solution set contains an extreme point of the closed convex hull of $\mathbf{F}$ which we denote by $H\mathbf{F}$. The set of extreme points of a family $\mathbf{F}$, we denote by $E\mathbf{F}$ and the set of support points by $\operatorname{supp} \mathbf{F}$.

We recall the definition of subordination between two functions, say f and F, analytic in Δ. This means that there is an analytic function ϕ so that $\phi \in B_0$ and $f = F \cdot \phi$. If F is univalent in Δ, the subordination is equivalent to $f(0) = F(0)$ and $f(\Delta) \subset F(\Delta)$. The set of all functions subordinate to a fixed function F we denote by $s(F)$. We note that $s(F)$ is a compact subset of $\mathbf{A}$ and $s(F) = \{F \cdot \phi\colon \phi \in B_0\}$.

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In Theorem 1 we give conditions that guarantee $\operatorname{supp} s(F) = \{F(xz) \colon |x| = 1\}$. *Support points of* $s(F)$.

THEOREM 1. *Let F be analytic in Δ and satisfy*

$$F(z) = G(z)/(z - x_0)^\alpha \tag{3}$$

where G is analytic in Δ, continuous in $\overline{\Delta}$, $G(x_0) \neq 0$ and $\alpha > 1$.

$$Hs(F) = \left\{ \int_{\partial\Delta} F(xz)\, d\mu(x) \colon \mu \in \Lambda \right\} \tag{4}$$

where Λ is the set of probability measures on $\partial\Delta$. Then $\operatorname{supp} s(F) = \{F(xz) \colon |x| = 1\}$.

PROOF. The inclusion $\{F(xz) \colon |x| = 1\} \subset \operatorname{supp} s(F)$ is known [**6**, p. 103] and so we need only to prove that $\operatorname{supp} s(F) \subset \{F(xz) \colon |x| = 1\}$.

Suppose that $f \in \operatorname{supp} s(F)$. It follows that $f = F \cdot \phi$ where $\phi \in \operatorname{supp} B_0$ [**1**, **5**]. There exists a continuous, linear functional J on $\mathbf{A}$ so that $\operatorname{Re} J$ is nonconstant on $s(F)$, J is given by $\{b_n\}$ as in (1) and (2) holds. Let $F(z) = A_0 + \sum_{n=1}^\infty A_n z^n$. Then $F(xz) = A_0 + \sum_{n=1}^\infty A_n x^n z^n$ and

$$J(F(xz)) = b_0 A_0 + \sum_{n=1}^\infty b_n A_n x^n = G(x) \tag{5}$$

defines a function G analytic in $\overline{\Delta}$, because $\overline{\lim}_{n\to\infty} \sqrt{|b_n A_n|} \leq \overline{\lim}_{n\to\infty} \sqrt{|b_n|} < 1$. Since (4) implies that $EHs(F) \subset \{F(xz) \colon |x| = 1\}$, it is clear that G is nonconstant, otherwise $\operatorname{Re} J$ is constant on $EHs(F)$ and so on $s(F)$. This is impossible by assumption. If we let $M = \max\{\operatorname{Re} J(g) \colon g \in s(F)\}$, then $\operatorname{Re} G(x) = M$ has only a finite number of solutions with $|x| = 1$ [**3**, p. 106]. By familiar arguments, [**3**, p. 100], we conclude that the solution set over $Hs(F)$ is the convex hull of a finite number of extreme points of $Hs(F)$. It follows that

$$f(z) = \sum_{k=1}^n \lambda_k F(x_k z) \tag{6}$$

where n is a positive integer, $\lambda_k \geq 0$, $|x_k| = 1$ $(k = 1, 2, \ldots, n)$ and $\sum_{k=1}^n \lambda_k = 1$. We see from (3) that $f(z) = G(\phi(z))/(\phi(z) - x_0)^\alpha$. Let $w_k = \bar{x}_k x_0$ $(k = 1, 2, \ldots, n)$ then (3), (6) and the previous equality imply $\phi(w_k) = x_0$ for $k = 1, 2, \ldots, n$. It is known [**4**, p. 83] that we can write

$$\frac{1}{1 - \bar{x}_0 \phi(z)} = \sum_{k=1}^m t_k \frac{1}{1 - y_k z} \tag{7}$$

where $t_k \geq 0$, $|y_k| = 1$ $(k = 1, 2, \ldots, m)$ and $\sum_{k=1}^m t_k = 1$. It follows that

$$\sum_{k=1}^n \lambda_k F(x_k z) = aG(\phi(z)) \left[\sum_{k=1}^m t_k \frac{1}{1 - y_k z} \right]^\alpha \tag{8}$$

where $a = (-x_0)^{-\alpha}$. By using (3) and comparing singularities on both sides of (8) we conclude that $n = m$ and so we have

$$\sum_{k=1}^n \lambda_k F(x_k z) = aG(\phi(z)) \left[\sum_{k=1}^n t_k \frac{1}{1 - y_k z} \right]^\alpha . \tag{9}$$

Using (3) we can rewrite (9) as

$$a\sum_{k=1}^{n}\lambda_k\frac{G(x_kz)}{(1-\bar{w}_kz)^\alpha}=aG(\phi(z))\left[\sum_{k=1}^{n}t_k\frac{1}{1-y_kz}\right]^\alpha. \tag{10}$$

Thus $w_k=\bar{y}_k$ for $k=1,2,\dots,n$. If we let $z=r\bar{y}_j$ ($j\in\{1,2,\dots,n\}$), multiply by $(1-r)^\alpha$ and let $r\to 1$ in (10) we obtain

$$G(x_j\bar{y}_j)\lambda_j=G(\phi(\bar{y}_j))t_j^\alpha. \tag{11}$$

But $\phi(\bar{y}_j)=\phi(w_j)=x_0=x_jw_j=x_j\bar{y}_j$ and $G(x_0)\neq 0$ by (3). Thus (11) implies $t_j^\alpha=\lambda_j$ for $j=1,2,\dots,n$.

To finish the proof, assume that each λ_k in (6) satisfies $0<\lambda_k<1$. As we showed above $\lambda_k=t_k^\alpha$ for $k=1,2,\dots,n$. Note that $\alpha>1$ and $0<t_k<1$ implies that $t_k^\alpha<t_k$ for $k=1,2,\dots,n$. Hence, we conclude that $1=\sum_{k=1}^n\lambda_k=\sum_{k=1}^n t_k^\alpha<\sum_{k=1}^n t_k=1$. This is a contradiction and so $\lambda_j=1$ for some j and $f(z)=F(x_jz)$. This completes the proof.

REMARK. In a recent paper [**2**], Y. Abu-Muhanna proved that if F is univalent in Δ, the complement of $F(\Delta)$ is convex and $\partial F(\Delta)$ satisfies a smoothness condition at ∞ then (3) holds. He also proved that under these assumptions $EHs(F)=\{F(xz)\colon |x|=1\}$ and so (4) also holds. Hence the conclusion of Theorem 1 holds under the assumptions made in [**2**]. When $F(z)=((1+cz)/(1-z))^\alpha$ for $|c|\le 1$, $c\neq -1$ and $\alpha>1$ it is known that $\operatorname{supp} s(F)=\{F(xz)\colon |x|=1\}$ [**5**]. This follows directly from Theorem 1.

COROLLARY 2. *Suppose F satisfies the hypothesis of Theorem* 1. *Suppose additionally that $F(z)\neq 0$ in Δ and $\log F(z)$ is convex. Then for $\beta\ge 1$,* $\operatorname{supp} s(F^\beta)=\{(F(xz))^\beta\colon |x|=1\}$.

PROOF. Note that F^β is analytic in Δ for some choice of $\log F(z)$. Also $(F(z))^\beta=K(z)/(z-x_0)^{\beta\alpha}$ where $K(z)=(G(z))^\beta$ is analytic in Δ, continuous in $\overline{\Delta}$, $K(x_0)\neq 0$ and $\beta\alpha>1$. It is easy to deduce from (4) and the fact that $\log F(z)$ is convex that $EHs(F^\beta)\subset\{(F(xz))^\beta\colon |x|=1\}$. It follows that (4) holds with F replaced by F^β. Hence the proof may be completed by appealing to Theorem 1.

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