SUPPORT POINTS OF SUBORDINATION FAMILIES

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ABSTRACT. Let \( s(F) \) denote the set of functions subordinate to a function \( F \) analytic in the unit disc \( \Delta \). Let \( Hs(F) \) denote the closed convex hull of \( s(F) \) and \( \text{supp}(s(F)) \) the set of support points of \( s(F) \). We prove the following

**Theorem.** Let \( F \) be analytic in \( \Delta \) and satisfy

1. \( Hs(F) = \{ \int_{\partial \Delta} F(xz)d\mu(z): \mu \text{ a probability measure on } \partial \Delta \} \)
2. \( F(z) = G(z)/(z - x_0)^\alpha \) where \( G \) is analytic in \( \Delta \), continuous in \( \overline{\Delta} \), \( G(x_0) \neq 0 \) and \( \alpha > 1 \). Then \( \text{supp}(s(F)) = \{ F(xz): |z| = 1 \} \).

**Introduction.** Let \( \Delta = \{ z: |z| < 1 \} \) and let \( \mathcal{A} \) denote the set of functions analytic in \( \Delta \). Let \( \mathcal{B}_0 = \{ f: f \in \mathcal{A} \text{ and } |f(z)| \leq |z| \} \). \( \mathcal{A} \) is a locally convex linear topological space with respect to the topology given by uniform convergence on compact subsets of \( \Delta \). By a continuous, linear functional on \( \mathcal{A} \), we mean a complex-valued functional defined on \( \mathcal{A} \) that is continuous and linear. In other words, if \( J \) is such a functional, then \( J(af + bg) = aJ(f) + bJ(g) \) whenever \( a \) and \( b \) belong to \( \mathbb{C} \) and \( f \) and \( g \) belong to \( \mathcal{A} \). Also, if \( f_n \in \mathcal{A} \) \( (n = 1, 2, \ldots) \) and \( f_n \to f \), then \( J(f_n) \to J(f) \). Each continuous, linear functional \( J \) on \( \mathcal{A} \) is given by a sequence \( \{ b_n \} \) \( (n = 1, 2, \ldots) \) which satisfies \( \lim_{n \to \infty} |b_n| < 1 \) and

\[
J(f) = \sum_{n=0}^{\infty} a_n b_n
\]

where \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) \( (|z| < 1) \) [7]. For such a sequence \( \{ b_n \} \), the function \( F(z) = \sum_{n=0}^{\infty} b_n z^n \) is analytic in \( \Delta = \{ z: |z| \leq 1 \} \).

A function \( f \) is called a support point of a compact subset \( F \) of \( \mathcal{A} \) if \( f \in F \) and if there is a continuous, linear functional \( J \) on \( \mathcal{A} \) so that

\[
\text{Re } J(f) = \max\{ \text{Re } J(g): g \in F \}
\]

and \( \text{Re } J \) is not constant on \( F \). In general, for a fixed continuous, linear functional \( J \), the solution set contains an extreme point of the closed convex hull of \( F \) which we denote by \( HF \). The set of extreme points of a family \( F \), we denote by \( EF \) and the set of support points by \( \text{supp} F \).

We recall the definition of subordination between two functions, say \( f \) and \( F \), analytic in \( \Delta \). This means that there is an analytic function \( \phi \) so that \( \phi \in B_0 \) and \( f = F \cdot \phi \). If \( F \) is univalent in \( \Delta \), the subordination is equivalent to \( f(0) = F(0) \) and \( f(\Delta) \subset F(\Delta) \). The set of all functions subordinate to a fixed function \( F \) we denote by \( s(F) \). We note that \( s(F) \) is a compact subset of \( \mathcal{A} \) and \( s(F) = \{ F \cdot \phi: \phi \in B_0 \} \).

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414
In Theorem 1 we give conditions that guarantee \( \text{supp}(s(F)) = \{F(xz) : |x| = 1\} \).

**Theorem 1.** Let \( F \) be analytic in \( \Delta \) and satisfy

\[ F(z) = G(z)/(z-x_0)^\alpha \]

where \( G \) is analytic in \( \Delta \), continuous in \( \overline{\Delta} \), \( G(x_0) \neq 0 \) and \( \alpha > 1 \).

\[ Hs(F) = \left\{ \int_{\partial\Delta} F(xz) \, d\mu(x) : \mu \in \Lambda \right\} \]

where \( \Lambda \) is the set of probability measures on \( \partial\Delta \). Then \( \text{supp}(s(F)) = \{F(xz) : |x| = 1\} \).

**Proof.** The inclusion \( \{F(xz) : |x| = 1\} \subseteq \text{supp}(s(F)) \) is known \([6, p. 103]\) and so we need only to prove that \( \text{supp}(s(F)) \subseteq \{F(xz) : |x| = 1\} \).

Suppose that \( f \in \text{supp}(s(F)) \). It follows that \( f = F \cdot \phi \) where \( \phi \in \text{supp}(B_0) \) \([1, 5]\).

There exists a continuous, linear functional \( J \) on \( A \) so that \( \text{Re } J \) is nonconstant on \( s(F) \), \( J \) is given by \( \{b_n\} \) as in (1) and (2) holds. Let \( F(z) = A_0 + \sum_{n=1}^{\infty} A_n z^n \). Then \( F(xz) = A_0 + \sum_{n=1}^{\infty} A_n x^n z^n \) and

\[ J(F(xz)) = b_0 A_0 + \sum_{n=1}^{\infty} b_n A_n x^n = G(x) \]

defines a function \( G \) analytic in \( \overline{\Delta} \), because \( \lim_{n \to \infty} \sqrt[n]{|b_n A_n|} \leq \lim_{n \to \infty} \sqrt[n]{|b_n|} < 1 \).

Since (4) implies that \( EHs(F) \subseteq \{F(xz) : |x| = 1\} \), it is clear that \( G \) is nonconstant, otherwise \( \text{Re } J \) is constant on \( EHs(F) \) and so on \( s(F) \). This is impossible by assumption. If we let \( M = \max\{\text{Re } J(g) : g \in s(F)\} \), then \( \text{Re } G(x) = M \) has only a finite number of solutions with \( |x| = 1 \) \([3, p. 106]\). By familiar arguments, \([3, p. 100]\), we conclude that the solution set over \( Hs(F) \) is the convex hull of a finite number of extreme points of \( Hs(F) \). It follows that

\[ f(z) = \sum_{k=1}^{n} \lambda_k F(x_k z) \]

where \( n \) is a positive integer, \( \lambda_k \geq 0 \), \( |x_k| = 1 \) for \( k = 1, 2, \ldots, n \) and \( \sum_{k=1}^{n} \lambda_k = 1 \). We see from (3) that \( f(z) = G(\phi(z))/\phi(z) - x_0)^\alpha \). Let \( w_k = \bar{x}_k x_0 \) \( (k = 1, 2, \ldots, n) \) then (3), (6) and the previous equality imply \( \phi(w_k) = x_0 \) for \( k = 1, 2, \ldots, n \). It is known \([4, p. 83]\) that we can write

\[ \frac{1}{1 - \bar{x}_0 \phi(z)} = \sum_{k=1}^{m} t_k \frac{1}{1 - y_k z} \]

where \( t_k \geq 0 \), \( |y_k| = 1 \) for \( k = 1, 2, \ldots, m \) and \( \sum_{k=1}^{m} t_k = 1 \). It follows that

\[ \sum_{k=1}^{n} \lambda_k F(x_k z) = a G(\phi(z)) \left[ \sum_{k=1}^{m} t_k \frac{1}{1 - y_k z} \right]^{\alpha} \]

where \( a = (-x_0)^{-\alpha} \). By using (3) and comparing singularities on both sides of (8) we conclude that \( n = m \) and so we have

\[ \sum_{k=1}^{n} \lambda_k F(x_k z) = a G(\phi(z)) \left[ \sum_{k=1}^{n} t_k \frac{1}{1 - y_k z} \right]^{\alpha} . \]
Using (3) we can rewrite (9) as

$$a \sum_{k=1}^{n} \lambda_k \frac{G(x_k z)}{(1 - \bar{w}_k z)^\alpha} = aG(\phi(z)) \left[ \sum_{k=1}^{n} \frac{1}{1 - \bar{y}_k z} \right]^\alpha.$$  

Thus $w_k = \bar{y}_k$ for $k = 1, 2, \ldots, n$. If we let $z = r\bar{y}_j$ ($j \in \{1, 2, \ldots, n\}$), multiply by $(1 - r)^\alpha$ and let $r \to 1$ in (10) we obtain

$$G(x_j \bar{y}_j) \lambda_j = G(\phi(\bar{y}_j)) t_j^\alpha.$$  

But $\phi(\bar{y}_j) = \phi(w_j) = x_0 = x_j w_j = x_j \bar{y}_j$ and $G(x_0) \neq 0$ by (3). Thus (11) implies $t_j^\alpha = \lambda_j$ for $j = 1, 2, \ldots, n$.

To finish the proof, assume that each $\lambda_k$ in (6) satisfies $0 < \lambda_k < 1$. As we showed above $\lambda_k = t_k^\alpha$ for $k = 1, 2, \ldots, n$. Note that $\alpha > 1$ and $0 < t_k < 1$ implies that $t_k^\alpha < t_k$ for $k = 1, 2, \ldots, n$. Hence, we conclude that $1 = \sum_{k=1}^{n} \lambda_k = \sum_{k=1}^{n} t_k^\alpha < \sum_{k=1}^{n} t_k = 1$. This is a contradiction and so $\lambda_j = 1$ for some $j$ and $f(z) = F(x_j z)$. This completes the proof.

**Remark.** In a recent paper [2], Y. Abu-Muhanna proved that if $F$ is univalent in $\Delta$, the complement of $F(\Delta)$ is convex and $\partial F(\Delta)$ satisfies a smoothness condition at $\infty$ then (3) holds. He also proved that under these assumptions $EHs(F) = \{F(xz) : |x| = 1\}$ and so (4) also holds. Hence the conclusion of Theorem 1 holds under the assumptions made in [2]. When $F(z) = ((1 + cz)/(1 - z))^a$ for $|c| \leq 1$, $c \neq -1$ and $\alpha > 1$ it is known that $\text{supp}(F) = \{F(xz) : |x| = 1\}$ [5]. This follows directly from Theorem 1.

**Corollary 2.** Suppose $F$ satisfies the hypothesis of Theorem 1. Suppose additionally that $F(z) \neq 0$ in $\Delta$ and $\log F(z)$ is convex. Then for $\beta \geq 1$, $\text{supp}(F^\beta) = \{(F(xz))^\beta : |x| = 1\}$.

**Proof.** Note that $F^\beta$ is analytic in $\Delta$ for some choice of $\log F(z)$. Also $(F^\beta(z)) = K(z)/(z - x_0)^{\beta a}$ where $K(z) = (G(z))^\beta$ is analytic in $\Delta$, continuous in $\Delta$, $K(x_0) \neq 0$ and $\beta a > 1$. It is easy to deduce from (4) and the fact that $\log F(z)$ is convex that $EHs(F^\beta) \subset \{(F(xz))^\beta : |x| = 1\}$. It follows that (4) holds with $F$ replaced by $F^\beta$. Hence the proof may be completed by appealing to Theorem 1.

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**References**


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