FUNCTIONS WHOSE DERIVATIVE HAS
POSITIVE REAL PART

R. R. LONDON

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ABSTRACT. In this paper we find a sharp upper bound for $|zf'(z)/f(z)|$, where $f$ is a normalised analytic function with $\text{Re } f'(z) > 0$ in the unit disc.

Denote by $R$ the class of functions $f$ which are regular in $D = \{z : |z| < 1\}$, and satisfy $f(0) = 0$, $f'(0) = 1$, $\text{Re } f'(z) > 0$ ($z \in D$). In a recent paper [3], D. K. Thomas proved, for some absolute constant $K$, that

$$zf'(z) < K(\log(1/(1-r)^{1-D}\log(U/d-r)))^{1/D}$$

whenever $f \in R$. He also asked what the sharp bound for $zf'/f$ might be, and it is this question that prompted the present paper. We prove

**Theorem.** Let $f \in R$. Then

$$\left|\frac{f'(z)}{f(z)}\right| \leq \frac{1+r}{(1-r)(-1-(2/r)\log(1-r))} \quad (0 < |z| = r < 1),$$

with equality for all $r$ in the case of the function $k \in R$ given by $k(z) = -z - 2\log(1-z)$.

Equality here, for $f = k$, occurs since

$$k'(r) = \frac{1+r}{1-r}, \quad \frac{k(r)}{r} = -1 - \frac{2}{r}\log(1-r) \quad (0 < r < 1),$$

and it is clear that the upper bound in the theorem is also a sharp upper bound for $|zf'(z)/f(z)|$, whenever $f \in R$.

**1. Proof of the theorem.** We shall use the following

**Lemma.** For $\rho$ and $t$ in $[0,1]$,

$$-\frac{8}{3}t^2\rho + (11\rho - 1)t^2 + 4(1-4\rho)t + 1 + \frac{11}{3}\rho \geq 0.$$

**Proof.** Denote the left side of the inequality by $g(\rho, t)$, and the square $[0,1] \times [0,1]$ by $S$. We have

$$g(\rho,0) = 1 + \frac{11}{3}\rho > 0, \quad g(\rho,1) = 4(1-\rho) \geq 0 \quad (0 \leq \rho \leq 1),$$

$$g(0,t) = -t^2 + 4t + 1 > 0, \quad g(1,t) = \frac{2}{3}(1-t)^2(7-4t) \geq 0 \quad (0 \leq t \leq 1),$$

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so that the minimum value of $g$ on $\partial S$ is zero. Next we consider the critical points $(p, t)$ of $g$, given by the equations

\begin{align*}
(2) & \quad 8t^3 - 33t^2 + 48t - 11 = 0, \\
(3) & \quad \rho(4t^2 - 11t + 8) + t - 2 = 0.
\end{align*}

The cubic in (2) is increasing, and so has just one real root $t$, which satisfies $.279 \leq t \leq .28$. The function $(2 - t)/(4t^2 - 11t + 8)$ is also increasing, and hence $.328 \leq \rho \leq .329$ when $\rho$ is given by (3) and $t$ by (2). By minimising each term of $g$ over these values of $\rho$ and $t$, we obtain $g(\rho, t) > 2$. Since the minimum value of $g$ is attained on $\partial S$, or at a critical point of $g$ inside $S$, the proof is now complete.

We can now prove the theorem. Let

$$k(z) = -z - 2 \log(1 - z),$$

then

$$k'(z) = \frac{1 + z}{1 - z},$$

which shows that $k \in R$. By putting $z = \rho e^{i\theta}$ and integrating with respect to $\rho$ over $[0, r]$, we obtain

$$\frac{k(z)}{z} = \frac{1}{r} \int_0^r \frac{1 + \rho e^{i\theta}}{1 - \rho e^{i\theta}} \, d\rho \quad (z = re^{i\theta}).$$

Now let $f \in R$, then $f'$ has a Herglotz representation [1, p. 22]

$$f'(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + e^{-it}z}{1 - e^{-it}z} \, d\mu(t),$$

which gives

$$\frac{f(z)}{z} = \frac{1}{2\pi} \int_0^{2\pi} \frac{k(z e^{-it})}{z e^{-it}} \, d\mu(t)$$

using (4), after a similar integration to the one above. Next let

$$\phi(r) = \max_{|z|=r} \left| \frac{1 + z}{1 - z} \right| \left/ \frac{\text{Re} \left( \frac{k(z)}{z} \right)}{z} \right. \quad (0 < r < 1),$$

and note that $\phi$ is well defined since $\text{Re}(k(z)/z) > 0$ ($z \in D$) by (4). Using (5) with (7), and then (6), we deduce

$$|f'(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{1 + e^{-it}z}{1 - e^{-it}z} \right| \, d\mu(t) \leq \frac{\phi(r)}{2\pi} \int_0^{2\pi} \text{Re} \left( \frac{k(z e^{-it})}{z e^{-it}} \right) \, d\mu(t) = \phi(r) \text{Re} \frac{f(z)}{z}.$$ 

Obviously, equality holds if $f = k$. To complete the proof we need to show that

$$\phi(r) = \frac{1 + r}{1 - r} \left/ \frac{k(r)}{r} \right.$$

and this we shall do by verifying that the maximum in (7) is attained at $z = r$.

We have

$$\frac{\partial}{\partial \theta} \left( \left| \frac{1 + z}{1 - z} \right| \left/ \frac{\text{Re} \left( \frac{k(z)}{z} \right)}{z} \right. \right)$$

$$= \left[ \left( \text{Re} \frac{k(z)}{z} \right) \left| \frac{1 + z}{1 - z} \right| \left( -\text{Im} \frac{2z}{1 - z^2} \right) \left| \frac{1 + z}{1 - z} \right| \frac{\partial}{\partial \theta} \left( \text{Re} \left( \frac{k(z)}{z} \right) \right) \right] \left/ \left( \text{Re} \left( \frac{k(z)}{z} \right) \right)^2 \right.$
where $z = re^{i\theta}$. Now using (4), with the notation

$$J(r, x) = 1 - 2rx + r^2, \quad x = \cos \theta,$$

we see that, apart from a nonnegative factor, the right side is

$$\left[- \left( \int_0^r \frac{1 - \rho^2}{J(\rho, x)} \, d\rho \right) \left( \frac{r}{J(r, x)} + \frac{r}{J(r, -x)} \right) + 2 \int_0^r \frac{\rho(1 - \rho^2)}{J^2(\rho, x)} \, d\rho \right] (\sin \theta).$$

So the maximum in (7) is attained at $z = r$ if the function in square brackets is nonpositive for $-1 \leq x \leq 1, 0 < r < 1$. If we write the corresponding inequality as

$$\int_0^r \frac{d}{dt} \left[ \int_0^t \frac{\rho(1 - \rho^2)}{J^2(\rho, x)} \, d\rho - \frac{1}{2} \left( \int_0^t \frac{1 - \rho^2}{J(\rho, x)} \, d\rho \right) \left( \frac{t}{J(t, x)} + \frac{t}{J(t, -x)} \right) \right] \, dt \leq 0,$$

the integrand is

$$\frac{1}{2} (1 - t^2) \left[ \frac{t}{J^2(t, x)} - \frac{t}{J(t, x)J(t, -x)} - \left( \frac{1}{J^2(t, x)} + \frac{1}{J^2(t, -x)} \right) \int_0^t \frac{1 - \rho^2}{J(\rho, x)} \, d\rho \right].$$

Thus it is sufficient to prove

$$\frac{1}{r} \int_0^r \frac{1 - \rho^2}{J(\rho, x)} \, d\rho \geq \frac{J(r, x) (J(r, -x) - J'(r, x))}{J^2(r, x) + J^2(r, -x)} = \frac{4rxJ(r, -x)}{J^2(r, x) + J^2(r, -x)},$$

where $-1 \leq x \leq 1, 0 < r < 1$. When $x$ is nonpositive this is obvious, so we now assume $0 < x < 1, 0 < r < 1$.

The last inequality can be written as

$$\frac{1}{r} \int_0^r \frac{1 - \rho^2}{J(\rho, x)} \, d\rho \geq \frac{1 - u}{1 + u^2},$$

where $u = J(r, x)/J(r, -x)$, and, since $0 < u < 1$, it is implied by

$$\frac{1}{r} \int_0^r \frac{1 - \rho^2}{J(\rho, x)} \, d\rho \geq 1 - u = \frac{4rx}{J(r, -x)}.$$

Now for $0 \leq \rho \leq x$ we have $0 < J(\rho, x) \leq 1 - \rho^2$, so that

$$(8) \quad \frac{1}{r} \int_0^r \frac{1 - \rho^2}{J(\rho, x)} \, d\rho \geq 1 \quad (0 < r \leq x),$$

and we have only to prove

$$(9) \quad \int_0^r \frac{1 - \rho^2}{J(\rho, x)} \, d\rho \geq \frac{4r^2x}{J(r, -x)} \quad (0 < x \leq r < 1).$$

Assume now that $0 < x \leq r < 1$. We have

$$\frac{d}{dr} \left( \frac{1}{J(r, x)} \right) = \frac{-2(r - x)}{J^2(r, x)} \leq 0,$$

and by using this with (8) we obtain

$$\int_0^r \frac{1 - \rho^2}{J(\rho, x)} \, d\rho \geq x + \int_x^r \frac{1 - \rho^2}{J(\rho, x)} \, d\rho$$

$$\geq x + \frac{1}{J(r, x)} \int_x^r (1 - \rho^2) \, d\rho$$

$$= x + \frac{1}{J(r, x)} \left( r - \frac{r^3}{3} - x + \frac{1}{3} x^3 \right).$$
So a proof of (9) is reduced to that of
\[
\frac{4r^2x}{J(r, -x)} \leq x + \frac{1}{J(r, x)} \left( r - \frac{1}{3}x^3 - x + \frac{1}{3}x^3 \right),
\]
which we observe is true for \(x = r\), since \(4rx < J(r, -x)\). The inequality is equivalent to
\[
\frac{1}{3}r^5 + \frac{1}{3}x^5 - (8x^2 + \frac{2}{3})r^3 + (\frac{11}{3}x^3 + x)r^2 + (-1 + 2x^2 - \frac{2}{3}x^4)r - \frac{1}{3}x^3 \leq 0,
\]
and because it is true for \(x = r\) it is implied by the inequality
\[
-\frac{1}{3}x^3r + (11r^2 - 1)x^2 + (4r - 16r^3)x + r^2 + \frac{1}{3}r^4 \geq 0,
\]
in which the left side is the derivative with respect to \(x\) of the left side of (10). If we now put \(x = tr\), and then \(\rho = r^2\), this inequality follows from the one in the lemma and the proof is complete.

2. Remarks. A general result of Ruscheweyh [2] has a direct bearing on the problem of maximising \(|zf'/f|\) for functions \(f\) in \(R\). It shows that the extreme function has the form
\[
f'(z) = t \frac{1 + \alpha z}{1 - \alpha z} + (1 - t) \frac{1 + \beta z}{1 - \beta z},
\]
where \(|\alpha| = |\beta| = 1, 0 < t \leq 1\). But in this approach the technical details seem rather more awkward than those given here.

Thomas [3] notes from (1) that for bounded functions \(f\) in \(R\)
\[
M(r, f') = O(1) \left[(1 - r) \log \frac{1}{1 - r}\right]^{-1},
\]
where \(M(r, f') = \max_{|z| = r} |f'(z)|\). The result of this paper shows that (11) also holds for \(f \in R\), whenever \(\text{Re}(f(z)/z)\) is bounded.

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REFERENCES