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INTERSECTIONS OF REFLECTIVE SUBCATEGORIES

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ABSTRACT. It is shown that the class of full reflective subcategories of $\mathcal{T}\!op$ (and of other concrete categories) is not closed under intersections. This answers a question raised by Herrlich in 1967. A natural example of nonreflective intersection is presented in the category of bitopological spaces.

We investigate reflective subcategories (subcategories are always understood full and replete) of concrete categories, i.e., categories equipped with a faithful functor to $\mathcal{S}et$. H. Herrlich asked whether each subcategory of $\mathcal{T}\!op$ has a reflective hull (or, equivalently, whether reflective subcategories of $\mathcal{T}\!op$ are closed under intersections) at the conference "Contributions to extension theory of topological structures" in Berlin, 1967; see also [$\mathbf{H}_1$]. The question has remained open in spite of considerable effort in the area; see [**T**] and references there. We present here a negative solution.

Recall from [**PT**] that a concrete category $\mathscr{K}$ is *universal* if each concrete category has a full embedding into $\mathscr{K}$. Further, $\mathscr{K}$ is *almost universal* if either it is universal, or if (a) constant maps between underlying sets always carry $\mathscr{K}$-morphisms, and (b) each concrete category $\mathscr{C}$ has an almost full embedding $E\colon \mathscr{C} \to \mathscr{K}$. This means that $E(\hom(C,C')) = \{f\colon EC \to EC' | f \text{ is nonconstant}\}$ for all objects C, C' of $\mathscr{C}$.

An object is said to have *rank* n for some regular cardinal n if the hom-functor of that object preserves n-directed colimits of extremal monos.

THEOREM. *Let $\mathscr{K}$ be a cocomplete, cowellpowered and almost universal concrete category in which each object has a rank. Then $\mathscr{K}$ has a collection of reflective subcategories with nonreflective intersection.*

PROOF. Morphisms in $\mathscr{K}$ have (epi, extremal mono-) factorizations (see [**HS**, 34.1]), and thus it follows from [**Ke**, 10.1] that for each morphism $f\colon A \to B$ the orthogonal subcategory $\{f\}^\perp$ is reflective. (Recall that $\{f\}^\perp$ consists of all K such that for each $p\colon A \to K$ there is a unique $q\colon B \to K$ with $p = $ q.f.) It is sufficient to find a class $\mathscr{H}$ of morphisms such that $\mathscr{H}^\perp = \bigcap_{f \in \mathscr{H}} \{f\}^\perp$ is nonreflective.

Define an auxiliary category $\mathscr{C}$. Its objects are A_i, B_i, and C for all ordinals i. Its morphisms are freely generated by the following morphisms $\alpha_{ij}\colon A_i \to A_j$ $(i, j \in \mathrm{Ord}, i < j)$, $\beta_{i,k}\colon A_i \to B_k$ $(i, k \in \mathrm{Ord})$, and $\gamma_i\colon C \to A_i$ $(i \in \mathrm{Ord})$, and the

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following relations:

$$\begin{aligned} \alpha_{ij} &= \alpha_{tj} \cdot \alpha_{it} && \text{for all } i < t < j, \\ \beta_{ik} &= \beta_{jk} \cdot \alpha_{ij} && \text{for all } i < j, \\ \beta_{ik} \cdot \gamma_i &= \beta_{kk} \cdot \gamma_k && \text{for all } k < i. \end{aligned}$$

$\mathscr{C}$ clearly satisfies the Isbell condition for concreteness (see [**A**]), and hence there is a full or almost full embedding $E\colon \mathscr{C} \to \mathscr{K}$. Put $\mathscr{H} = \{E\alpha_{0i}\}_{i\in\mathrm{Ord}}$. Assuming that EA_0 has a reflection $r\colon EA_0 \to R$ in $\mathscr{H}^\perp$, we shall derive a contradiction by showing that $\hom(EC, R)$ is large.

Each EB_k lies in $\mathscr{H}^\perp$: if E is full, this follows from $\hom(EA_i, EB_k) = \{E\beta_{ik}\}$, and if E is almost full, use $\hom(EA_i, EB_k) = \{E\beta_{ik}\} \cup \{f\colon EA_i \to EB_k | f \text{ is constant}\}$. Thus, for each $k \in \mathrm{Ord}$ we have $d_k\colon R \to EB_k$ with $E\beta_{0k} = d_k \cdot r$. Since $R \in \mathscr{H}^\perp$, for each $i \in \mathrm{Ord}$ we have $a_i\colon EA_i \to R$ with $r = a_i \cdot E\alpha_{0i}$. We conclude that

$$E\beta_{ik} \cdot E\alpha_{0i} = E\beta_{0k} = d_k \cdot a_i \cdot E\alpha_{0i} \qquad (i, k \in \mathrm{Ord}),$$

and since $EB_k \in \mathscr{H}^\perp$, it follows that $E\beta_{ik} = d_k \cdot a_i$. For $i < k$ we have $\beta_{ik} \cdot \gamma_i \neq \beta_{kk} \cdot \gamma_k$ in $\mathscr{C}$, and thus, $d_k \cdot a_i \cdot E\gamma_i \neq d_k \cdot a_k \cdot E\gamma_k$. We see that the morphisms $a_i \cdot E\gamma_i\colon EC \to R$ ($i \in \mathrm{Ord}$) are pairwise distinct—a contradiction.

COROLLARY. *The category $\mathscr{T\!op}$ has a collection of reflective subcategories with nonreflective intersection.*

In fact, the almost universality of $\mathscr{T\!op}$ is established in [**Ko**], and the remaining hypotheses are easy to verify.

REMARK. Assuming (M) (the nonexistence of a proper class of measurable cardinals), an impressive collection of categories are known to be almost universal; see [**PT**]. This includes the categories of metric spaces, posets, semigroups, rings, each of which also satisfies the remaining hypotheses of the above Theorem.

EXAMPLE. In the category $\mathscr{B\!itop}$ of bitopological spaces, consider the subcategories 1-$\mathscr{C\!omp}$ and 2-$\mathscr{C\!omp}$ of all spaces in which the first and second topology, respectively, is compact and T_2. Then 1-$\mathscr{C\!omp}$ is reflective: form the Čech-Stone compactification of the first topology while the second one is final w.r.t. the reflection map; analogously, 2-$\mathscr{C\!omp}$ is reflective. However,

$$\mathscr{B\!icomp} = 1\text{-}\mathscr{C\!omp} \cap 2\text{-}\mathscr{C\!omp}$$

is a nonreflective subcategory.

PROOF. For an infinite discrete topological space (X, d), assume that (X, d, d) has a reflection R in $\mathscr{B\!icomp}$; we derive a contradiction by constructing arbitrarily large spaces B_i in $\mathscr{B\!icomp}$, each of which contains X as a dense subset (i.e., no proper subset of B_i containing X is closed in both topologies). Then the embedding of X to B_i factors through a continuous map from R which is clearly surjective, and hence, $\operatorname{card} R \geq \operatorname{card} B_i$—a contradiction.

Define bitopological spaces $A_i = (X_i, p_i, q_i)$ by induction. First, $A_o = (X, d, d)$. Given A_i with i even, put $A_{i+1} = (\beta X_i, \beta p_i, q_{i+1})$, where $(\beta X_i, \beta p_i)$ is the Čech-Stone compactification of (X_i, p_i), and q_{i+1} is the extension of q_i for which $\beta X_i - X_i$ is a discrete summand; analogously, for i odd, put $A_{i+1} = (\beta X_i, p_{i+1}, \beta q_i)$. For limit ordinals, $A_i = \bigcup_{j<i} A_j$ with both topologies final. The spaces $B_i = A_i \cup \{\infty\}$

obtained by simultaneous one-point compactification of both topologies of A_i lie in $\mathcal{Bicomp}$, and X is dense in each of them. To verify that they are arbitrarily large, we will show that $X_{i+1} - X_i$ is infinite for each i. This is clear for isolated ordinals i since $X_{i+1} = \beta X_i$ and X_i has an infinite discrete summand (in the topology on which the compactification is performed). For a limit ordinal i, consider all nets $u\colon i \to X_i$ with $u_j \in X_{j+1} - X_j$ for $j < i$ odd, and $u_j \in X_{j+2} - X_{j+1}$ for $j < i$ even. Each net u has an accumulation point $x_u \in \beta X_i - X_i$, and if such nets u and v are disjoint, then $x_u \neq x_v$ (since for each $k \leq i + 1$ there is a continuous map $g\colon (X_k, p_k) \to [0,1]$ with $g(u_j) = 0$ and $g(v_j) = 1$ for all $j < k$; this is easy to prove by induction on k). Thus, $X_{i+1} - X_i$ is infinite.

REMARK. Note that $\mathcal{Bitop}$ is an extremely well-behaved category: it is topological over $\mathcal{Set}$ (see [$\mathbf{H}_2$]) and hence, complete, cocomplete, wellpowered and cowellpowered. In the category $\mathcal{Bitop}\ T_2$ of bitopological spaces with both topologies T_2, the categories 1-$\mathcal{Comp}$ and 2-$\mathcal{Comp}$ are even epireflective, with nonreflective intersection. Also the category $\mathcal{Bitop}\ T_2$ is rather well-behaved, being monotopological over $\mathcal{Set}$; however, it is not cowellpowered.

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